$$\begin{aligned}
Y_{i,j}(Q) &= (E_{i,n+j} - E_{j,n+i})(\textstyle\sum (e_p \wedge e_{n+p}))\\
&= 2 \cdot e_j \wedge e_i\\
&\neq 0
\end{aligned}$$

so that whenever $a < i, j \leq n - b$,

$$\begin{aligned}
&Y_{i,j}(w^{(a,b)})\\
&= Y_{i,j}(e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b)/2})\\
&= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Y_{i,j}((\textstyle\sum (e_p \wedge e_{n+p}))^{(k-a-b)/2})\\
&= (k-a-b) \cdot (e_1 \wedge \cdots \wedge e_a \wedge e_j \wedge e_i \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b-2)/2})\\
&\neq 0.
\end{aligned}$$

It is always possible to find a pair (i, j) satisfying the conditions $a < i, j \leq n - b$ since we are assuming $a + b < k < n$; this concludes the proof of part (i).

The proof of part (ii) requires only one further step: we have to check the vectors $w^{(a,b)}$ with $a + b = k = n$ to see if any of them might be highest weight vectors for $\mathfrak{so}_{2n}\mathbb{C}$. In fact (as the statement of the theorem implies), two of them are: It is not hard to check that, in fact, $w^{(n,0)}$ and $w^{(n-1,1)}$ are killed by every positive root space $\mathfrak{g}_{L_i+L_j}$. To see that no other vector $w^{(a,n-a)}$ is, look at the action of $Y_{a+1,a+2} \in \mathfrak{g}_{L_{a+1}+L_{a+2}}$: we have

$$\begin{aligned}
&Y_{a+1,a+2}(w^{(a,n-a)})\\
&= (E_{a+1,n+a+2} - E_{a+2,n+a+1})(e_1 \wedge \cdots \wedge e_a \wedge e_{n+a+1} \wedge \cdots \wedge e_{2n})\\
&= e_1 \wedge \cdots \wedge e_a \wedge e_{a+1} \wedge e_{n+a+1} \wedge e_{n+a+3} \wedge \cdots \wedge e_{2n}\\
&\quad - e_1 \wedge \cdots \wedge e_a \wedge e_{a+2} \wedge e_{n+a+2} \wedge \cdots \wedge e_{2n}\\
&\neq 0.
\end{aligned}$$

□

Remarks. (i) This theorem will be a consequence of the Weyl character formula, which will tell us a priori that the dimension of the irreducible representation of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weight $L_1 + \cdots + L_k$ has dimension $\binom{2n}{k}$ if $k < n$, and half that if $k = n$.

(ii) Note also that by the above, $\wedge^n V$ is the direct sum of the two irreducible representations $\Gamma_{2\alpha}$ and $\Gamma_{2\beta}$ with highest weights $2\alpha = L_1 + \cdots + L_n$ and $2\beta = L_1 + \cdots + L_{n-1} - L_n$. Indeed, the inclusion $\Gamma_{2\alpha} \oplus \Gamma_{2\beta} \subset \wedge^n V$ can be seen just from the weight diagram: $\wedge^n V$ possesses a highest weight vector with highest weight $L_1 + \cdots + L_n$, and so contains a copy of $\Gamma_{2\alpha}$; but this representation does not possess the weight 2β, and so $\wedge^n V$ must contain $\Gamma_{2\beta}$ as well. (Alternatively, we observed in the preceding lecture that in choosing an ordering of the roots we could have chosen our linear functional $l = c_1 H_1 + \cdots + c_n H_n$ with $c_1 > c_2 > \cdots > -c_n > 0$ without altering the positive roots or the Weyl chamber; in this case the weight λ of $\wedge^n V$ with $l(\lambda)$ maximal would be 2β, showing that $\Gamma_{2\beta} \subset \wedge^n V$.)

(iii) If we want to avoid weight diagrams altogether, we can still see that $\wedge^n V$ must be reducible, because the action of $\mathfrak{so}_{2n}\mathbb{C}$ preserves two bilinear forms: first, we have the bilinear form induced on $\wedge^n V$ by the form Q on V; and second we have the wedge product

$$\varphi\colon \wedge^n V \times \wedge^n V \to \wedge^{2n} V = \mathbb{C},$$

the last map taking $e_1 \wedge \cdots \wedge e_{2n}$ to 1. It follows that $\wedge^n V$ is reducible; indeed, if we want to see the direct sum decomposition asserted in the statement of the theorem we can look at the composition

$$\tau\colon \wedge^n V \to \wedge^n V^* \to \wedge^n V,$$

where the first map is the isomorphism given by Q and the second is the isomorphism given by φ. The square of this map is the identity, and decomposing $\wedge^n V$ into $+1$ and -1 eigenspaces for this map gives two subrepresentations.

Exercise 19.3*. Part (i) of Theorem 19.2 can also be proved by showing that for any nonzero vector $w \in \wedge^k V$, the linear span of the vectors $X(w)$, for $X \in \mathfrak{so}_m\mathbb{C}$, is all of $\wedge^k V$. For these purposes take, instead of the basis we have been using, an orthonormal basis $v_1, \ldots, v_m$ for $V = \mathbb{C}^m$, $m = 2n$, so $Q(v_i, v_j) = \delta_{i,j}$. The vectors $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$, $I = \{i_1 < \cdots < i_k\}$, form a basis for $\wedge^k V$, and $\mathfrak{so}_m\mathbb{C}$ has a basis consisting of endomorphisms $V_{p,q}$, $p < q$, which takes v_q to v_p, v_p to $-v_q$, and takes the other v_i to zero. Compute the images $V_{p,q}(v_I)$, and prove the claim, first, when $w = v_I$ for some I, and then by induction on the number of nonzero coefficients in the expression $w = \sum a_I v_I$. For (ii) a similar argument shows that $\wedge^n V$ is an irreducible representation of the group $\mathrm{O}_n\mathbb{C}$, and the ideas of §5.1 (cf. §19.5) can be used to see how it decomposes over the subgroup $\mathrm{SO}_n\mathbb{C}$ of index two.

We return now to our analysis of the representations of $\mathfrak{so}_{2n}\mathbb{C}$. By the theorem, the exterior powers $V, \wedge^2 V, \ldots, \wedge^{n-2} V$ provide us with the irreducible representations with highest weight the fundamental weight along the first $n - 2$ edges of the Weyl chamber (of course, the exterior power $\wedge^{n-1} V$ is irreducible as well, but as we have observed, $L_1 + \cdots + L_{n-1}$ is not on an edge of the Weyl chamber, and so $\wedge^{n-1} V$ is not as useful for our purposes). For the remaining two edges, we have found irreducible representations with highest weights located there, namely the two direct sum factors of $\wedge^n V$; but the highest weights of these two representations are not primitive ones; they are divisible by 2. Thus, given the theorem above, we see that we have constructed exactly one-half the irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$, namely, those whose highest weight lies in the sublattice $\mathbb{Z}\{L_1, \ldots, L_n\} \subset \Lambda_W$. Explicitly, any weight γ in the closed Weyl chamber can be expressed (uniquely) in the form

$$\gamma = a_1 L_1 + \cdots + a_{n-2}(L_1 + \cdots + L_{n-2})$$
$$+ a_{n-1}(L_1 + \cdots + L_{n-1} - L_n)/2 + a_n(L_1 + \cdots + L_n)/2$$

with $a_i \in \mathbb{N}$. If $a_{n-1} + a_n$ is even, with $a_{n-1} \geq a_n$ we see that the representation

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2} V) \otimes \mathrm{Sym}^{a_n}(\wedge^{n-1} V) \otimes \mathrm{Sym}^{(a_{n-1}-a_n)/2}(\Gamma_{2\beta})$$

will contain an irreducible representation Γ_γ with highest weight γ; whereas if $a_n \geq a_{n-1}$, we will find Γ_γ inside

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2} V) \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V) \otimes \mathrm{Sym}^{(a_n-a_{n-1})/2}(\Gamma_{2\alpha}).$$

There remains the problem of constructing irreducible representations Γ_γ whose highest weight γ involves an odd number of α's and β's. To do this, we clearly have to exhibit irreducible representations Γ_α and Γ_β with highest weights α and β. These exist, and are called the *spin representations* of $\mathfrak{so}_{2n}\mathbb{C}$; we will study them in detail in the following lecture. We see from the above that once we exhibit the two representations Γ_α and Γ_β, we will have constructed all the representations of $\mathfrak{so}_{2n}\mathbb{C}$. The representation Γ_γ with highest weight γ written above will be found in the tensor product

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2} V) \otimes \mathrm{Sym}^{a_{n-1}}(\Gamma_\beta) \otimes \mathrm{Sym}^{a_n}(\Gamma_\alpha).$$

For the time being, we will assume the existence of the spin representations of $\mathfrak{so}_{2n}\mathbb{C}$; there is a good deal we can say about these representations just on the basis of their weight diagrams.

Exercise 19.4*. Find the weights (with multiplicities) of the representations $\wedge^k V$, and also of $\Gamma_{2\alpha}$, $\Gamma_{2\beta}$, Γ_α, and Γ_β.

Exercise 19.5. Using the above, show that Γ_α and Γ_β are dual to one another when n is odd, and that they are self-dual when n is even.

Exercise 19.6. Give the complete decomposition into irreducible representations of $\mathrm{Sym}^2\Gamma_\alpha$ and $\wedge^2\Gamma_\alpha$. Show that

$$\Gamma_\alpha \otimes \Gamma_\alpha = \Gamma_{2\alpha} \oplus \wedge^{n-2} V \oplus \wedge^{n-4} V \oplus \wedge^{n-6} V \otimes \cdots.$$

Exercise 19.7. Show that

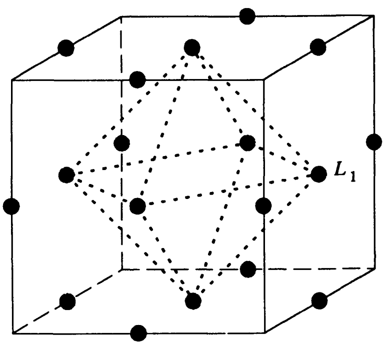
$$\Gamma_\alpha \otimes \Gamma_\beta = \wedge^{n-1} V \oplus \wedge^{n-3} V \oplus \wedge^{n-5} V \oplus \cdots.$$

Exercise 19.8. Verify directly the above statements in the case of $\mathfrak{so}_6\mathbb{C}$, using the isomorphism with $\mathfrak{sl}_4\mathbb{C}$.

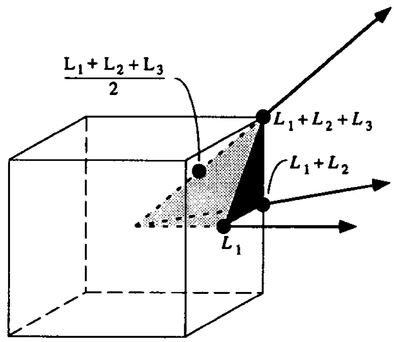
Exercise 19.9. Show that the automorphism of $\mathbb{C}^{2n}$ that interchanges e_n and e_{2n}, leaving the other e_i fixed, determines an automorphism of $\mathfrak{so}_{2n}\mathbb{C}$ that preserves the $n-2$ roots $L_1 - L_2, \ldots, L_{n-2} - L_{n-1}$ and interchanges $L_{n-1} - L_n$ and $L_{n-1} + L_n$. This automorphism takes the representation V to itself, but interchanges Γ_α and Γ_β.

§19.3. Representations of $\mathfrak{so}_7\mathbb{C}$

While we might reasonably be apprehensive about the prospect of a family of Lie algebras even more strangely behaved than the even orthogonal algebras, there is some good news: even though the roots systems of the odd Lie algebras appear more complicated than those of the even, the representation theory of the odd algebras is somewhat tamer. We will describe these representations, starting with the example of $\mathfrak{so}_7\mathbb{C}$; we begin, as always, with a picture of the root diagram:

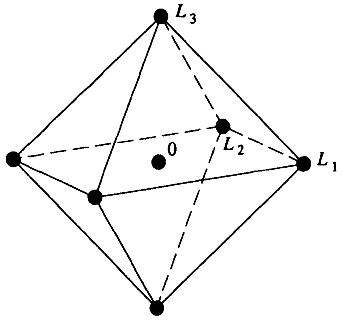


As we said, this looks like the root diagram for $\mathfrak{sp}_6\mathbb{C}$, except that the roots $\pm 2L_i$ have been shortened to $\pm L_i$. Unlike the case of $\mathfrak{so}_5\mathbb{C}$, however, where the long and short roots could be confused and the root diagram was correspondingly congruent to that of $\mathfrak{sp}_4\mathbb{C}$, in the present circumstance the root diagram is not similar to any other; the Lie algebra $\mathfrak{so}_7\mathbb{C}$, in fact, is *not* isomorphic to any of the others we have studied. Next, the Weyl chamber:



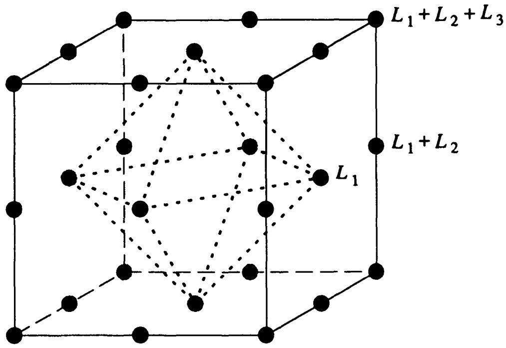
Again, the Weyl chamber itself looks just like that of $\mathfrak{sp}_6\mathbb{C}$; the difference in this picture is in the weight lattice, which contains the additional vector $(L_1 + L_2 + L_3)/2$.

As usual, we start our study of the representations of $\mathfrak{so}_7\mathbb{C}$ with the standard representation, whose weights are $\pm L_i$ and 0:



Note that the highest weight L_1 of this representation lies along the front edge of the Weyl chamber. Next, the weights of the exterior square $\wedge^2 V$ are $\pm L_i \pm L_j$, $\pm L_i$, and 0 (taken three times); this, of course, is just the adjoint representation. Note that the highest weight $L_1 + L_2$ of this representation is the same as that of the exterior square of the standard representation for $\mathfrak{so}_6\mathbb{C}$, but because of the smaller Weyl chamber this weight does indeed lie on an edge of the chamber.

Next, consider the third exterior power $\wedge^3 V$ of the standard. This has weights $\pm L_1 \pm L_2 \pm L_3$, $\pm L_i \pm L_j$, $\pm L_i$ (with multiplicity 2) and 0 (with multiplicity 3), i.e., at the midpoints of all the vertices, edges, and faces of the cube:



It is not obvious, from the weight diagram alone, that this is an irreducible representation; it could be that $\wedge^3 V$ contains a copy of the standard representation V and that the irreducible representation $\Gamma_{L_1+L_2+L_3}$ thus has multiplicity 1 on the weights $\pm L_i$ and multiplicity 2 (or 1) at 0. We can rule out this possibility by direct calculation: for example, if this were the case, then $\wedge^3 V$ would contain a highest weight vector with weight L_1. The weight space with eigenvalue L_1 in $\bigwedge^3 V$ is spanned by the tensors $e_1 \wedge e_2 \wedge e_5$ and $e_1 \wedge e_3 \wedge e_6$, however, and if we apply to these the generators $X_{1,2} = E_{1,2} - E_{5,4}$, $X_{2,3} = E_{2,3} - E_{6,5}$, and $U_3 = E_{3,7} - E_{7,6}$ of the root spaces corresponding to the positive roots $L_1 - L_2$, $L_2 - L_3$, and L_3, we see that

$$X_{2,3}(e_1 \wedge e_3 \wedge e_6) = e_1 \wedge e_2 \wedge e_6,$$

$$U_3(e_1 \wedge e_3 \wedge e_6) = e_1 \wedge e_3 \wedge e_7 \neq 0;$$

$$X_{2,3}(e_1 \wedge e_2 \wedge e_5) = e_1 \wedge e_2 \wedge e_6,$$

$$U_3(e_1 \wedge e_2 \wedge e_5) = 0.$$

There is thus no linear combination of $e_1 \wedge e_2 \wedge e_5$ and $e_1 \wedge e_3 \wedge e_6$ killed by both U_3 and $X_{2,3}$, showing that $\bigwedge^3 V$ has no highest weight vector of weight L_1.

Exercise 19.10. Verify that $\bigwedge^3 V$ does not contain the trivial representation.

We have thus found irreducible representations of $\mathfrak{so}_7\mathbb{C}$ with highest weight vectors along the three edges of the Weyl chamber, and as in the case of $\mathfrak{so}_6\mathbb{C}$ we have thereby established the existence of the irreducible representations of $\mathfrak{so}_7\mathbb{C}$ with highest weight in the sublattice $\mathbb{Z}\{L_1, L_2, L_3\}$. To complete the description, we need to know that the representation Γ_α with highest weight $\alpha = (L_1 + L_2 + L_3)/2$ exists, and what it looks like, and this time there is no isomorphism to provide this; we will have to wait until the following lecture. In the meantime, we can still have fun playing around both with the representations we do know exist, and also with those whose existence is simply asserted.

Exercise 19.11. Find the decomposition into irreducible representations of the tensor product $V \otimes \bigwedge^2 V$; in particular find the multiplicities of the irreducible representation $\Gamma_{2L_1+L_2}$ with highest weight $2L_1 + L_2$.

Exercise 19.12. Show that the symmetric square of the representation Γ_α decomposes into a copy of $\bigwedge^3 V$ and a trivial one-dimensional representation.

Exercise 1913. Find the decomposition into irreducible representations of $\bigwedge^2 \Gamma_\alpha$.

§19.4. Representations of the Odd Orthogonal Algebras

We will now describe as much as we can of the general pattern for representations of the odd orthogonal Lie algebras $\mathfrak{so}_{2n+1}\mathbb{C}$. As in the case of the even orthogonal Lie algebras, the proof of the existence part of the basic theorem (14.18) (that is, the construction of the irreducible representation with given highest weight) will not be complete until the following lecture, but we can work around this pretty well.

To begin with, recall that the weight lattice of $\mathfrak{so}_{2n+1}\mathbb{C}$ is, like that of $\mathfrak{so}_{2n}\mathbb{C}$, generated by $L_1, \ldots, L_n$ together with the further vector $(L_1 + \cdots + L_n)/2$. The Weyl chamber, on the other hand, is the cone

$$\mathscr{W} = \{\sum a_i L_i: a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.$$

The Weyl chamber is as we have pointed out the same as for $\mathfrak{sp}_{2n}\mathbb{C}$, that is, it is a simplicial cone with faces corresponding to the n planes $a_1 = a_2, \ldots, a_{n-1} = a_n$ and $a_n = 0$. The edges of the Weyl chamber are thus the rays generated by the vectors $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1}$ and $L_1 + \cdots + L_n$ (note that $L_1 + \cdots + L_{n-1}$ *is* on an edge of the Weyl chamber). Again, the intersection of the weight lattice with the closed Weyl cone is a free semigroup, in this case generated by the fundamental weights $\omega_1 = L_1$, $\omega_2 = L_1 + L_2$, $\ldots$, $\omega_{n-1} = L_1 + \cdots + L_{n-1}$ and the weight $\omega_n = \alpha = (L_1 + \cdots + L_n)/2$. Moreover, as we saw in the cases of $\mathfrak{so}_5\mathbb{C}$ and $\mathfrak{so}_7\mathbb{C}$, the exterior powers of the standard representation do serve to generate all the irreducible representations whose highest weights are in the sublattice $\mathbb{Z}\{L_1, \ldots, L_n\}$: in general we have the following theorem.

Theorem 19.14. *For $k = 1, \ldots, n$, the exterior power $\bigwedge^k V$ of the standard representation V of $\mathfrak{so}_{2n+1}\mathbb{C}$ is the irreducible representation with highest weight $L_1 + \cdots + L_k$.*

PROOF. We will leave this as an exercise; the proof is essentially the same as in the case of $\mathfrak{so}_{2n}\mathbb{C}$, with enough of a difference to make it interesting. □

We have thus constructed one-half of the irreducible representations of $\mathfrak{so}_{2n+1}\mathbb{C}$: any weight γ in the closed Weyl chamber can be written

$$\gamma = a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1}) + a_n(L_1 + \cdots + L_n)/2$$

with $a_i \in \mathbb{N}$; and if a_n is even, the representation

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\textstyle\bigwedge^{n-1} V) \otimes \mathrm{Sym}^{a_n/2}(\textstyle\bigwedge^n V)$$

will contain an irreducible representation Γ_γ with highest weight γ. We are still missing, however, any representation whose weights involve odd multiples of α; to construct these, we clearly have to exhibit an irreducible representation Γ_α with highest weight α. This exists and is called (as in the case of the even orthogonal Lie algebras) the *spin representation* of $\mathfrak{so}_{2n+1}\mathbb{C}$. We see from the above that once we exhibit the spin representation Γ_α, we will have constructed all the representations of $\mathfrak{so}_{2n+1}\mathbb{C}$; for any γ as above the tensor

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\textstyle\bigwedge^{n-1} V) \otimes \mathrm{Sym}^{a_n}(\Gamma_\alpha)$$

will contain a copy of Γ_γ.

As in the case of the spin representation Γ_α of the even orthogonal Lie algebras, we can say some things about Γ_α even in advance of its explicit construction; for example, we can do the following exercises.

Exercise 19.15. Find the weights (with multiplicities) of the representations $\bigwedge^k V$, and also of Γ_α.

Exercise 19.16. Give the complete decomposition into irreducible representations of $\mathrm{Sym}^2\Gamma_\alpha$ and $\bigwedge^2\Gamma_\alpha$. Show that

$$\Gamma_\alpha \otimes \Gamma_\alpha = \textstyle\bigwedge^n V \oplus \bigwedge^{n-1} V \oplus \bigwedge^{n-2} V \oplus \cdots \oplus \bigwedge^1 V \oplus \bigwedge^0 V.$$

Exercise 19.17. Verify directly the above statements in the case of $\mathfrak{so}_5\mathbb{C}$, using the isomorphism with $\mathfrak{sp}_4\mathbb{C}$.

§19.5. Weyl's Construction for Orthogonal Groups

The same procedure we saw in the symplectic case can be used to construct representations of the orthogonal groups, this time generalizing what we saw directly for $\bigwedge^k V$ in §§19.2 and 19.4. For the symmetric form Q on $V = \mathbb{C}^m$, the same formula (17.9) determines contractions from $V^{\otimes d}$ to $V^{\otimes(d-2)}$. Denote the intersection of the kernels of all these contractions by $V^{[d]}$. For any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$ of d, let

$$\mathbb{S}_{[\lambda]} V = V^{[d]} \cap \mathbb{S}_\lambda V. \tag{19.18}$$

As before, this is a representation of the orthogonal group $\mathrm{O}_m\mathbb{C}$ of Q.

Theorem 19.19. *The space $\mathbb{S}_{[\lambda]} V$ is an irreducible representation of $\mathrm{O}_m\mathbb{C}$; $\mathbb{S}_{[\lambda]} V$ nonzero if and only if the sum of the lengths of the first two columns of the Young diagram of λ is at most m.*

The tensor power $V^{\otimes d}$ decomposes exactly as in Lemma 17.15, with everything the same but replacing the symbol $\langle d \rangle$ by $[d]$. In particular,

$$\mathbb{S}_{[\lambda]} V = V^{[d]} \cdot c_\lambda = \mathrm{Im}(c_\lambda \colon V^{[d]} \to V^{[d]}).$$

Exercise 19.20. Verify that $\mathbb{S}_{[\lambda]} V$ is zero when the sum of the lengths of the first two columns is greater than m by showing that $\bigwedge^a V \otimes \bigwedge^b V \otimes V^{(d-a-b)}$ is contained in $\sum_I \Psi_I(V^{\otimes(d-2)})$ when $a + b > m$. Show that $\mathbb{S}_{[\lambda]} V$ is not zero when the sum of the lengths of the first two columns is at most m.

Exercise 19.21*. (i) Show that the kernel of the contraction from $\mathrm{Sym}^d V$ to $\mathrm{Sym}^{d-2} V$ is the irreducible representation $\mathbb{S}_{[d]} V$ of $\mathfrak{so}_m\mathbb{C}$ with highest weight dL_1.

(ii) Show that

$$\mathrm{Sym}^d V = \mathbb{S}_{[d]} V \oplus \mathbb{S}_{[d-2]} V \oplus \cdots \oplus \mathbb{S}_{[d-2p]} V,$$

where p is the largest integer $\leq d/2$.

The proof of the theorem proceeds exactly as in §17.3. The fundamental fact from invariant theory is the same statement as (17.19), with, of course, the operators $\vartheta_I = \Psi_I \circ \Phi_I$ defined using the given symmetric form, and the group $Sp_{2n}\mathbb{C}$ replaced by $O_m\mathbb{C}$ (and the same reference to Appendix F.2 for the proof). The theorem then follows from Lemma 6.22 in exactly the same way as for the symplectic group.

To find the irreducible representations over $SO_m\mathbb{C}$ one can proceed as in §5.1. Weyl calls two partitions (each with the sum of the first two column lengths at most m) *associated* if the sum of the lengths of their first columns is m and the other columns of their Young diagrams have the same lengths. Representations of associated partitions restrict to isomorphic representations of $SO_m\mathbb{C}$. Note that at least one of each pair of associated partitions will have a Young diagram with at most $\frac{1}{2}m$ rows. If $m = 2n + 1$ is odd, no λ is associated to itself, but if $m = 2n$ is even, any λ with a Young diagram with n nonzero rows will be associated to itself, and its restriction will be the sum of two conjugate representations of $SO_m\mathbb{C}$ of the same dimension. The final result is:

Theorem 19.22. (i) *If $m = 2n + 1$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, then $\mathbb{S}_{[\lambda]}V$ is the irreducible representation of $\mathfrak{so}_m\mathbb{C}$ with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.*

(ii) *If $m = 2n$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0)$, then $\mathbb{S}_{[\lambda]}V$ is the irreducible representation of $\mathfrak{so}_m\mathbb{C}$ with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.*

(iii) *If $m = 2n$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n > 0)$, then $\mathbb{S}_{[\lambda]}V$ is the sum of two irreducible representations of $\mathfrak{so}_m\mathbb{C}$ with highest weights $\lambda_1 L_1 + \cdots + \lambda_n L_n$ and $\lambda_1 L_1 + \cdots + \lambda_{n-1} L_{n-1} - \lambda_n L_n$.*

Exercise 19.23. When m is odd, show that $O_m\mathbb{C} = SO_m\mathbb{C} \times \{\pm I\}$. Show that if λ and μ are associated, then $\mu = \lambda \otimes \varepsilon$, where ε is the sign of the determinant.

We postpone to Lecture 25 all discussion of multiplicities of weight spaces, or decomposing tensor products or restrictions to subgroups.

As we saw in Lecture 15 for $GL_n\mathbb{C}$ and in Lecture 17 for $Sp_{2n}\mathbb{C}$, it is possible to make a commutative algebra $\mathbb{S}^{[\cdot]} = \mathbb{S}^{[\cdot]}(V)$ out of the sum of all the irreducible representations of $SO_m\mathbb{C}$, where $V = \mathbb{C}^m$ is the standard representation. First suppose $m = 2n + 1$ is odd. Define the ring $\mathbb{S}^\cdot(V, n)$ as in §15.5, which is a sum of all the representations $\mathbb{S}_\lambda(V)$ of $GL(V)$ where λ runs over all partitions with at most n parts. As in the symplectic case, there is a canonical decomposition

$$\mathbb{S}_\lambda(V) = \mathbb{S}_{[\lambda]}(V) \oplus J_{[\lambda]}(V),$$

and the direct sum $J^{[\cdot]} = \bigoplus_\lambda J_{[\lambda]}(V)$ is an ideal in $\mathbb{S}^\cdot(V, n)$. The quotient ring

$$\mathbb{S}^{[\cdot]}(V) = A^\cdot(V, n)/J^{[\cdot]} = \bigoplus_\lambda \mathbb{S}_{[\lambda]}(V)$$

is a commutative graded ring which contains each irreducible representation of $SO_{2n+1}\mathbb{C}$ once.

If $m = 2n$ is even, the above quotient will contain each representation $\mathbb{S}_{[\lambda]}(V)$ twice if λ has n rows. To cut it down so there is only one of each, one can add to $J^{[\cdot]}$ relations of the form $x - \tau(x)$, for $x \in \bigwedge^n V$, where $\tau: \bigwedge^n V \to \bigwedge^n V$ is the isomorphism described in the remark (iii) after the proof of Theorem 19.2. For a detailed discussion, with explicit generators for the ideas, see [L-T].

LECTURE 20

Spin Representations of $\mathfrak{so}_m\mathbb{C}$

In this lecture we complete the picture of the representations of the orthogonal Lie algebras by constructing the spin representations $S^{\pm}$ of $\mathfrak{so}_m\mathbb{C}$; this also yields a description of the spin groups $\mathrm{Spin}_m\mathbb{C}$. Since the representation-theoretic analysis of the spaces $S^{\pm}$ was carried out in the preceding lecture, we are concerned here primarily with the algebra involved in their construction. Thus, §20.1 and §20.2, while elementary, involve some fairly serious algebra. Section 20.3, where we briefly sketch the notion of triality, may seem mysterious to the reader (this is at least in part because it is so to the authors); if so, it may be skipped. Finally, we should say that the subject of the spin representations of $\mathfrak{so}_m\mathbb{C}$ is a very rich one, and one that accommodates many different points of view; the reader who is interested is encouraged to try some of the other approaches that may be found in the literature.

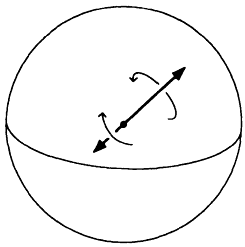
§20.1. Clifford Algebras and Spin Representations of $\mathfrak{so}_m\mathbb{C}$

We begin this section by trying to motivate the definition of Clifford algebras. We may begin by asking, why were we able to find all the representations of $\mathrm{SL}_n\mathbb{C}$ or $\mathrm{Sp}_{2n}\mathbb{C}$ inside tensor powers of the standard representation, but only half the representations of $\mathrm{SO}_m\mathbb{C}$ arise this way? One difference that points in this direction lies in the topology of these groups: $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$ are simply connected, while $\mathrm{SO}_m\mathbb{C}$ has fundamental group $\mathbb{Z}/2$ for $m > 2$ (for proofs see §23.1). Therefore $\mathrm{SO}_m\mathbb{C}$ has a double covering, the *spin group* $\mathrm{Spin}_m\mathbb{C}$. (For $m \leq 6$, these coverings could also be extracted from our identifications of the adjoint group $\mathrm{PSO}_m\mathbb{C}$ with the adjoint group of other simply connected groups; e.g. the double cover of $\mathrm{SO}_3\mathbb{C}$ is $\mathrm{SL}_2\mathbb{C}$.) We will see that the missing representations are those representations of $\mathrm{Spin}_m\mathbb{C}$ that do not come from representations of $\mathrm{SO}_m\mathbb{C}$.

This double covering may be most readily visible, and probably familiar, for the case of the real subgroup $\mathrm{SO}_3\mathbb{R}$ of rotations: a rotation is specified by an axis to rotate about, given by a unit vector u, and an angle of rotation about u; the two choices $\pm u$ of unit vector give a two-sheeted covering. In other words, if D^3 is the unit ball in $\mathbb{R}^3$, there is a double covering

$$S^3 = D^3/\partial D^3 \to \mathrm{SO}_3\mathbb{R},$$

which sends a vector v in D^3 to rotation by the angle $2\pi\|v\|$ about the unit vector $v/\|v\|$ (the origin and the unit sphere ∂D^3 are sent to the identity transformation).



This covering is even easier to see for the entire orthogonal group $\mathrm{O}_3\mathbb{R}$, which is generated by reflections R_v in unit vectors v (with $\pm v$ determining the same reflection): we can describe the double cover of $\mathrm{O}_3\mathbb{R}$ as the group generated by unit vectors v, with relations

$$v_1 \cdot \ldots \cdot v_n = w_1 \cdot \ldots \cdot w_m$$

whenever the compositions of the corresponding reflections are equal, i.e., whenever

$$R_{v_1} \circ \cdots \circ R_{v_n} = R_{w_1} \circ \cdots \circ R_{w_m};$$

and also relations

$$(-v)\cdot(-w) = v \cdot w$$

for all pairs of unit vectors v and w. (Note that if we restricted ourselves to products of even numbers of the generators $v \in \partial D^3$ we would get back the double cover of the special orthogonal group $\mathrm{SO}_3\mathbb{C}$.)

How should we generalize this? The answer is not obvious. For one thing, for various reasons we will not try to construct directly a group that covers the orthogonal group in general. Instead, given a vector space V (real or complex) and a quadratic form Q on V, we will first construct an algebra $\mathrm{Cliff}(V, Q)$, called the *Clifford algebra*. The algebra $\mathrm{Cliff}(V, Q)$ will then turn out to contain in its multiplicative group a subgroup which is a double cover of the orthogonal group $O(V, Q)$ of automorphisms of V preserving Q.

By analogy with the construction of the double cover of $SO_3\mathbb{R}$, the Clifford algebra $\mathrm{Cliff}(V, Q)$ associated to the pair (V, Q) is an associative algebra containing and generated by V. (When we want to describe the spin group inside $\mathrm{Cliff}(V, Q)$ we will restrict ourselves to products of even numbers of elements of V having a fixed norm $Q(v, v)$; if odd products are allowed as well, we get a group called "Pin" which is a double covering of the whole orthogonal group.) To motivate the definition, we would like $\mathrm{Cliff}(V, Q)$ to be the algebra generated by V subject to relations analogous to those above for the double cover of the orthogonal group. In particular, for any vector v with $Q(v, v) = 1$, since the reflection R_v in the hyperplane perpendicular to v is an involution, we want

$$v \cdot v = 1$$

in $\mathrm{Cliff}(V, Q)$. By polarization, this is the same as imposing the relation

$$v \cdot w + w \cdot v = 2Q(v, w)$$

for all v and w in V. In particular, $w \cdot v = -v \cdot w$ if v and w are perpendicular. In fact, the Clifford algebra[1] will be defined below to be the associative algebra generated by V and subject to the equation $v \cdot v = Q(v, v)$.

Looking ahead, we will see later in this section that each complex Clifford algebra contains an orthogonal Lie algebra as a subalgebra. The key theorem is then that $\mathrm{Cliff}(V, Q)$ *is isomorphic either to a matrix algebra or to a sum of two matrix algebras*. This in turn determines either one or two representations of the orthogonal Lie algebras, which turn out to be the representations which were needed to complete the story in the last lecture. Just as in the special linear and symplectic cases, the corresponding Lie groups are not really needed to construct the representations; they can be written down directly from the Lie algebra. In this section we do this, using the Clifford algebras to construct these representations of $\mathfrak{so}_m\mathbb{C}$ directly, and verify that they give the missing spin representations. In the second section of this lecture we will show how the spin groups sit as subgroups in their multiplicative groups.

Clifford Algebras

Given a symmetric bilinear form Q on a vector space V, the *Clifford algebra* $C = C(Q) = \mathrm{Cliff}(V, Q)$ is an associative algebra with unit 1, which contains and is generated by V, with $v \cdot v = Q(v, v) \cdot 1$ for all $v \in V$. Equivalently, we have the equation

$$v \cdot w + w \cdot v = 2Q(v, w), \tag{20.1}$$

[1] The mathematical world seems to be about evenly divided about the choice of signs here, and one must translate from Q to $-Q$ to go from one side to the other.

for all v and w in V. The Clifford algebra can be defined to be the universal algebra with this property: if E is any associative algebra with unit, and a linear mapping $j\colon V \to E$ is given such that $j(v)^2 = Q(v, v)\cdot 1$ for all $v \in V$, or equivalently

$$j(v)\cdot j(w) + j(w)\cdot j(v) = 2Q(v, w)\cdot 1 \tag{20.2}$$

for all $v, w \in V$, then there should be a unique homomorphism of algebras from $C(Q)$ to E extending j. The Clifford algebra can be constructed quickly by taking the tensor algebra

$$T^{\bullet}(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{C} \oplus V \oplus (V\otimes V) \oplus (V\otimes V\otimes V) \oplus \cdots,$$

and setting $C(Q) = T^{\bullet}(V)/I(Q)$, where $I(Q)$ is the two-sided ideal generated by all elements of the form $v\otimes v - Q(v, v)\cdot 1$. It is automatic that this $C(Q)$ satisfies the required universal property.

The facts that the dimension of C is 2^m, where $m = \dim(V)$, and that the canonical mapping from V to C is an embedding, are part of the following lemma:

Lemma 20.3. *If $e_1, \ldots, e_m$ form a basis for V, then the products $e_I = e_{i_1}\cdot e_{i_2}\cdot\ldots\cdot e_{i_k}$, for $I = \{i_1 < i_2 < \cdots < i_k\}$, and with $e_\phi = 1$, form a basis for $C(Q) = \mathrm{Cliff}(V, Q)$.*

Proof. From the equations $e_i\cdot e_j + e_j\cdot e_i = 2Q(e_i, e_j)$ it follows immediately that the elements e_I generate $C(Q)$. Their independence is not hard to verify directly; it also follows by seeing that the images in the matrix algebras under the mappings constructed below are independent. For another proof, note that when $Q \equiv 0$, the Clifford algebra is just the exterior algebra $\bigwedge^{\bullet} V$. In general, the Clifford algebra can be filtered by subspaces F_k, consisting of those elements which can be written as sums of at most k products of elements in V; one checks that the associated graded space F_k/F_{k+1} is $\bigwedge^k V$. For a third proof, one can verify that the Clifford algebra of the direct sum of two orthogonal spaces is the skew commutative tensor product of the Clifford algebras of the two spaces (cf. Exercise B.9), which reduces one to the trivial case where $\dim V = 1$. $\square$

Since the ideal $I(Q) \subset T(V)$ is generated by elements of even degree, the Clifford algebra inherits a $\mathbb{Z}/2\mathbb{Z}$ grading:

$$C = C^{\mathrm{even}} \oplus C^{\mathrm{odd}} = C^+ \oplus C^-,$$

with $C^+\cdot C^+ \subset C^+$, $C^+\cdot C^- \subset C^-$, $C^-\cdot C^+ \subset C^-$, $C^-\cdot C^- \subset C^+$; C^+ is spanned by products of an even number of elements in V and C^- is spanned by products of an odd number. In particular, C^{even} is a subalgebra of dimension 2^{m-1}.

Since $C(Q)$ is an associative algebra, it determines a Lie algebra, with bracket $[a, b] = a\cdot b - b\cdot a$. From now on we assume Q is nondegenerate. The new representations of $\mathfrak{so}_m\mathbb{C}$ will be found in two steps:

(i) embedding the Lie algebra $\mathfrak{so}(Q) = \mathfrak{so}_m\mathbb{C}$ inside the Lie algebra of the even part of the Clifford algebra $C(Q)$;
(ii) identifying the Clifford algebras with one or two copies of matrix algebras.

To carry out the first step we make explicit the isomorphism of $\bigwedge^2 V$ with $\mathfrak{so}(Q)$ that we have discussed before. Recall that

$$\mathfrak{so}(Q) = \{X \in \operatorname{End}(V): Q(Xv, w) + Q(v, Xw) = 0 \text{ for all } v, w \text{ in } V\}.$$

The isomorphism is given by

$$\textstyle\bigwedge^2 V \xrightarrow{\cong} \mathfrak{so}(Q) \subset \operatorname{End}(V), \qquad a \wedge b \mapsto \varphi_{a\wedge b},$$

for a and b in V, where $\varphi_{a\wedge b}$ is defined by

$$\varphi_{a\wedge b}(v) = 2(Q(b, v)a - Q(a, v)b). \tag{20.4}$$

It is a simple verification that $\varphi_{a\wedge b}$ is in $\mathfrak{so}(Q)$. One sees that the natural bases correspond up to scalars, e.g., $e_i \wedge e_{n+j}$ maps to $2(E_{i,j} - E_{n+j,n+i})$, so the map is an isomorphism. (The choice of scalar factor is unimportant here; it was chosen to simplify later formulas.) One calculates what the bracket on $\bigwedge^2 V$ must be to make this an isomorphism of Lie algebras:

$$\begin{aligned}
[\varphi_{a\wedge b}, \varphi_{c\wedge d}](v) &= \varphi_{a\wedge b} \circ \varphi_{c\wedge d}(v) - \varphi_{c\wedge d} \circ \varphi_{a\wedge b}(v) \\
&= 2\varphi_{a\wedge b}(Q(d, v)c - Q(c, v)d) - 2\varphi_{c\wedge d}(Q(b, v)a - Q(a, v)b) \\
&= 4Q(d, v)(Q(b, c)a - Q(a, c)b) \\
&\quad - 4Q(c, v)(Q(b, d)a - Q(a, d)b) \\
&\quad - 4Q(b, v)(Q(d, a)c - Q(c, a)d) \\
&\quad + 4Q(a, v)(Q(d, b)c - Q(c, b)d) \\
&= 2Q(b, c)\varphi_{a\wedge d}(v) - 2Q(b, d)\varphi_{a\wedge c}(v) \\
&\quad - 2Q(a, d)\varphi_{c\wedge b}(v) + 2Q(a, c)\varphi_{d\wedge b}(v).
\end{aligned}$$

This gives an explicit formula for the bracket on $\bigwedge^2 V$:

$$\begin{aligned}
[a \wedge b, c \wedge d] &= 2Q(b, c)a \wedge d - 2Q(b, d)a \wedge c \\
&\quad - 2Q(a, d)c \wedge b + 2Q(a, c)d \wedge b.
\end{aligned} \tag{20.5}$$

On the other hand, the bracket in the Clifford algebra satisfies

$$\begin{aligned}
[a\cdot b, c\cdot d] &= a\cdot b\cdot c\cdot d - c\cdot d\cdot a\cdot b \\
&= (2Q(b, c)a\cdot d - a\cdot c\cdot b\cdot d) - (2Q(a, d)c\cdot b - c\cdot a\cdot d\cdot b) \\
&= 2Q(b, c)a\cdot d - (2Q(b, d)a\cdot c - a\cdot c\cdot d\cdot b) \\
&\quad - 2Q(a, d)c\cdot b + (2Q(a, c)\cdot d\cdot b - a\cdot c\cdot d\cdot b) \\
&= 2Q(b, c)a\cdot d - 2Q(b, d)a\cdot c - 2Q(a, d)c\cdot b + 2Q(a, c)\cdot d\cdot b.
\end{aligned}$$

It follows that the map $\psi\colon \bigwedge^2 V \to \mathrm{Cliff}(V, Q)$ defined by

$$\psi(a \wedge b) = \tfrac{1}{2}(a \cdot b - b \cdot a) = a \cdot b - Q(a, b) \tag{20.6}$$

is a map[2] of Lie algebras, and by looking at basis elements again one sees that it is an embedding. This proves:

Lemma 20.7. *The mapping $\psi \circ \varphi^{-1}\colon \mathfrak{so}(Q) \to C(Q)^{\mathrm{even}}$ embeds $\mathfrak{so}(Q)$ as a Lie subalgebra of $C(Q)^{\mathrm{even}}$.*

Exercise 20.8. Show that the image of ψ is

$$F_2 \cap C(Q)^{\mathrm{even}} \cap \mathrm{Ker}(\mathrm{trace}),$$

where F_2 is the subspace of $C(Q)$ spanned by products of at most two elements of V, and the trace of an element of $C(Q)$ is the trace of left multiplication by that element on $C(Q)$.

We consider first the *even* case: write $V = W \oplus W'$, where W and W' are n-dimensional isotropic spaces for Q. (Recall that a space is isotropic when Q restricts to the zero form on it.) With our choice of standard Q on $V = \mathbb{C}^{2n}$, W can be taken to be the space spanned by the first n basis vectors, W' by the last n.

Lemma 20.9. *The decomposition $V = W \oplus W'$ determines an isomorphism of algebras*

$$C(Q) \cong \mathrm{End}(\textstyle\bigwedge^{\bullet} W),$$

where $\bigwedge^{\bullet} W = \bigwedge^0 W \oplus \cdots \oplus \bigwedge^n W$.

Proof. Mapping $C(Q)$ to the algebra $E = \mathrm{End}(\bigwedge^{\bullet} W)$ is the same as defining a linear mapping from V to E, satisfying (20.2). We must construct maps $l\colon W \to E$ and $l'\colon W' \to E$ such that

$$l(w)^2 = 0, \qquad l'(w')^2 = 0, \tag{20.10}$$

and

$$l(w) \circ l'(w') + l'(w') \circ l(w) = 2Q(w, w')I$$

for any $w \in W$, $w' \in W'$. For each $w \in W$, let $L_w \in E$ be left multiplication by w on the exterior algebra $\bigwedge^{\bullet} W$:

$$L_w(\xi) = w \wedge \xi, \quad \xi \in \textstyle\bigwedge^{\bullet} W.$$

For $\vartheta \in W^*$, let $D_\vartheta \in E$ be the derivation of $\bigwedge^{\bullet} W$ such that $D_\vartheta(1) = 0$, $D_\vartheta(w) = \vartheta(w) \in \bigwedge^0 W = \mathbb{C}$ for $w \in W = \bigwedge^1 W$, and

[2] Note that the bilinear form ψ given by (20.6) is alternating since $\psi(a \wedge a) = 0$, so it defines a linear map on $\bigwedge^2 V$.

$$D_\vartheta(\zeta \wedge \xi) = D_\vartheta(\zeta) \wedge \xi + (-1)^{\deg(\zeta)}\zeta \wedge D_\vartheta(\zeta).$$

Explicitly, $D_\vartheta(w_1 \wedge \cdots \wedge w_r) = \sum(-1)^{i-1}\vartheta(w_i)(w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_r)$. Now set

$$l(w) = L_w, \qquad l'(w') = D_\vartheta, \tag{20.11}$$

where $\vartheta \in W^*$ is defined by the identity $\vartheta(w) = 2Q(w, w')$ for all $w \in W$. The required equations (20.10) are straightforward verifications: one checks directly on elements in $W = \bigwedge^1 W$, and then that, if they hold on ζ and ξ, they hold on $\zeta \wedge \xi$. Finally, one may see that the resulting map is an isomorphism by looking at what happens to a basis. □

Exercise 20.12. The left $C(Q)$-module $\bigwedge^\bullet W$ is isomorphic to a left ideal in $C(Q)$. Show that if f is a generator for $\bigwedge^n W'$, then $C(Q)\cdot f = \bigwedge^\bullet W \cdot f$, and the map $\zeta \mapsto \zeta \cdot f$ gives an isomorphism

$$\textstyle\bigwedge^\bullet W \to \bigwedge^\bullet W \cdot f = C(Q)\cdot f$$

of left $C(Q)$-modules.

Now we have a decomposition $\bigwedge^\bullet W = \bigwedge^{\text{even}} W \oplus \bigwedge^{\text{odd}} W$ into the sum of even and odd exterior powers, and $C(W)^{\text{even}}$ respects this splitting. We deduce from Lemma 20.9 an isomorphism

$$C(Q)^{\text{even}} \cong \operatorname{End}(\textstyle\bigwedge^{\text{even}} W) \oplus \operatorname{End}(\bigwedge^{\text{odd}} W). \tag{20.13}$$

Combining with Lemma 20.7, we now have an embedding of Lie algebras:

$$\mathfrak{so}(Q) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\textstyle\bigwedge^{\text{even}} W) \oplus \mathfrak{gl}(\bigwedge^{\text{odd}} W), \tag{20.14}$$

and hence we have two representations of $\mathfrak{so}(Q) = \mathfrak{so}_{2n}\mathbb{C}$, which we denote by

$$S^+ = \textstyle\bigwedge^{\text{even}} W \quad \text{and} \quad S^- = \bigwedge^{\text{odd}} W.$$

Proposition 20.15. *The representations $S^\pm$ are the irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weights $\alpha = \frac{1}{2}(L_1 + \cdots + L_n)$ and $\beta = \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$. More precisely,*

$$S^+ = \Gamma_\alpha \quad \textit{and} \quad S^- = \Gamma_\beta \quad \textit{if } n \textit{ is even};$$

$$S^+ = \Gamma_\beta \quad \textit{and} \quad S^- = \Gamma_\alpha \quad \textit{if } n \textit{ is odd}.$$

Proof. We show that the natural basis vectors $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $\bigwedge^\bullet W$ are weight vectors. Tracing through the isomorphisms established above, we see that $H_i = E_{i,i} - E_{n+i,n+i}$ in $\mathfrak{h} \subset \mathfrak{so}_{2n}\mathbb{C}$ corresponds to $\frac{1}{2}(e_i \wedge e_{n+i})$ in $\bigwedge^2 V$, which corresponds to $\frac{1}{2}(e_i \cdot e_{n+i} - 1)$ in $C(Q)$, which maps to

$$\tfrac{1}{2}(L_{e_i} \circ D_{2e_i^*} - I) = L_{e_i} \circ D_{e_i^*} - \tfrac{1}{2}I \in \operatorname{End}(\textstyle\bigwedge^\bullet W).$$

A simple calculation shows that

$$L_{e_i} \circ D_{e_i^*}(e_I) = \begin{cases} e_I & \text{if } i \in I \\ 0 & \text{if } i \notin I. \end{cases}$$

Therefore, e_I spans a weight space with weight $\frac{1}{2}(\sum_{i \in I} L_i - \sum_{j \notin I} L_j)$. All such weights with given $|I|$ mod 2 are congruent by the Weyl group, so each of $S^+ = \bigwedge^{\text{even}} W^+$ and $S^- = \bigwedge^{\text{odd}} W$ must be an irreducible representation. The highest weights are easy to read off. For example, the highest weight for $\bigwedge^{\text{even}} W$ is $\frac{1}{2}\sum L_i = \alpha$ if n is even, while if n is odd, its highest weight is β. □

These two representations S^+ and S^- are usually called the *half-spin representations* of $\mathfrak{so}_{2n}\mathbb{C}$, while their sum $S = S^+ \oplus S^- = \bigwedge^{\bullet} W$ is called the *spin representation.* Frequently, especially when we speak of the even and odd cases together, we call them all simply "spin representations." Elements of S are called *spinors.* For other proofs of the proposition see Exercises 20.34 and 20.35.

For the *odd* case, write $V = W \oplus W' \oplus U$, where W and W' are n-dimensional isotropic subspaces, and U is a one-dimensional space perpendicular to them. For our standard Q on $\mathbb{C}^{2n+1}$, these are spanned by the first n, the second n, and the last basis vector.

Lemma 20.16. *The decomposition $V = W \oplus W' \oplus U$ determines an isomorphism of algebras*

$$C(Q) \cong \operatorname{End}(\textstyle\bigwedge^{\bullet} W) \oplus \operatorname{End}(\bigwedge^{\bullet} W').$$

Proof. Proceeding as in the even case, to map V to $E = \operatorname{End}(\bigwedge^{\bullet} W)$, map $w \in W$ to L_w, $w' \in W'$ to D_ϑ, where $\vartheta(w) = 2Q(w, w')$ as before. Let u_0 be the element in U such that $Q(u_0, u_0) = 1$, and send u_0 to the endomorphism that is the identity on $\bigwedge^{\text{even}} W$, and minus the identity on $\bigwedge^{\text{odd}} W$. Since this involution skew commutes with all L_w and D_ϑ, the resulting map from $V = W \oplus W' \oplus U$ to E determines an algebra homomorphism from $C(Q)$ to E. The map to $\operatorname{End}(\bigwedge^{\bullet} W')$ is defined similarly, reversing the roles of W and W'. Again one checks that the map is an isomorphism by looking at bases. □

Exercise 20.17*. Find a generator for a left ideal of $C(Q)$ that is isomorphic to $\bigwedge^{\bullet} W$.

The subalgebra $C(Q)^{\text{even}}$ of $C(Q)$ is mapped isomorphically onto either of the factors by the isomorphism of the lemma, so we have an isomorphism in the odd case:

$$C(Q)^{\text{even}} \cong \operatorname{End}(\textstyle\bigwedge^{\bullet} W). \tag{20.18}$$

As before, this gives a representation $S = \bigwedge^{\bullet} W$ of Lie algebras:

$$\mathfrak{so}_{2n+1}\mathbb{C} = \mathfrak{so}(Q) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\textstyle\bigwedge^{\bullet} W) = \mathfrak{gl}(S). \tag{20.19}$$

Proposition 20.20. *The representation $S = \bigwedge^{\bullet} W$ is the irreducible representation of $\mathfrak{so}_{2n+1}\mathbb{C}$ with highest weight*

$$\alpha = \tfrac{1}{2}(L_1 + \cdots + L_n).$$

PROOF. Exactly as in the even case, each e_I is an eigenvector with with weight $\frac{1}{2}(\sum_{i\in I} L_i - \sum_{j\notin I} L_j)$. This time all such weights are congruent by the Weyl group, so this must be an irreducible representation, and the highest weight is clearly $\frac{1}{2}(L_1 + \cdots + L_n)$. □

As we saw in Lecture 19, the construction of this *spin representation S* finishes the proof of the existence theorem for representations of $\mathfrak{so}_m\mathbb{C}$, and hence for all of the classical complex semisimple Lie algebras.

Exercise 20.21*. Use the above identification of the Clifford algebras with matrix algebras (or direct calculation) to compute their centers. In particular, show that the intersection of the center of C with the even subalgebra C^{even} is always the one-dimensional space of scalars. Show similarly that if x is in C^{odd} and $x \cdot v = -v \cdot x$ for all v in V, then $x = 0$.

Exercise 20.22*. For $X \in \mathfrak{so}(Q)$ and $v \in V$, we have $X \cdot v \in V$ by the standard action of $\mathfrak{so}(Q)$ on V. On the other hand, we have identified $\mathfrak{so}(Q)$ and V as subspaces of the Clifford algebra C, so we can compute the commutator $[X, v]$. Show that these agree:

$$X \cdot v = [X, v] \in V \subset C.$$

Problem 20.23*. Let $C(p, q)$ be the real Clifford algebra corresponding to the quadratic form with p positive and q negative eigenvalues. Lemmas 20.9 and 20.16 actually construct isomorphisms of $C(n, n)$ with a real matrix algebra, and of $C(n + 1, n)$ with a product of two real matrix algebras. Compute $C(p, q)$ for other p and q. All are products of one or two matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

§20.2. The Spin Groups $\mathrm{Spin}_m\mathbb{C}$ and $\mathrm{Spin}_m\mathbb{R}$

The Clifford algebra $C = C(Q)$ is generated by the subspace $V = \mathbb{C}^m$, and C has an anti-involution $x \mapsto x^*$, determined by

$$(v_1 \cdot \ldots \cdot v_r)^* = (-1)^r v_r \cdot \ldots \cdot v_1$$

for any $v_1, \ldots, v_r$ in V. This operation $*$, sometimes called the *conjugation*, is the composite of:

the *main antiautomorphism* or reversing map τ: $C \to C$ determined by

$$\tau(v_1 \cdot \ldots \cdot v_r) = v_r \cdot \ldots \cdot v_1 \tag{20.24}$$

for $v_1, \ldots, v_r$ in V, and

the *main involution* α which is the identity on C^{even} and minus the identity on C^{odd}, i.e.,

$$\alpha(v_1 \cdot \ldots \cdot v_r) = (-1)^r v_1 \cdot \ldots \cdot v_r. \tag{20.25}$$

Note that $(x \cdot y)^* = y^* \cdot x^*$, which comes from the identities $\tau(x \cdot y) = \tau(y) \cdot \tau(x)$ and $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$.

Exercise 20.26. Use the universal property for C to verify that these are well defined: show that α is a homomorphism from C to C and τ is a well-defined homomorphism from C to the opposite algebra of C (the algebra with the same vector space structure, but with reversed multiplication: $x \mathbin{\tilde{\cdot}} y = y \cdot x$).

Instead of defining the spin group as the set of products of certain elements of V, it will be convenient to start with a more abstract definition. Set

$$\operatorname{Spin}(Q) = \{x \in C(Q)^{\text{even}} : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}. \tag{20.27}$$

We see from this definition that $\operatorname{Spin}(Q)$ forms a closed subgroup of the group of units in the (even) Clifford algebra. Any x in $\operatorname{Spin}(Q)$ determines an endomorphism $\rho(x)$ of V by

$$\rho(x)(v) = x \cdot v \cdot x^*, \quad v \in V.$$

Proposition 20.28. *For $x \in \operatorname{Spin}(Q)$, $\rho(x)$ is in $\mathrm{SO}(Q)$. The mapping*

$$\rho\colon \operatorname{Spin}(Q) \to \mathrm{SO}(Q)$$

is a homomorphism, making $\operatorname{Spin}(Q)$ a connected two-sheeted covering of $\mathrm{SO}(Q)$. The kernel of ρ is $\{1, -1\}$.

Proof. We will prove something more. Define a larger subgroup, this time of the multiplicative group of $C(Q)$, by

$$\operatorname{Pin}(Q) = \{x \in C(Q) : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}, \tag{20.29}$$

and define a homomorphism

$$\rho\colon \operatorname{Pin}(Q) \to \mathrm{O}(Q), \qquad \rho(x)(v) = \alpha(x) \cdot v \cdot x^*, \tag{20.30}$$

where $\alpha\colon C(Q) \to C(Q)$ is the main involution.

To see that $\rho(x)$ preserves the quadratic form Q, we use the fact that for w in V, $Q(w, w) = w \cdot w = -w \cdot w^*$, and calculate:

$$\begin{aligned} Q(\rho(x)(v), \rho(x)(v)) &= -\alpha(x) \cdot v \cdot x^* \cdot (\alpha(x) \cdot v \cdot x^*)^* \\ &= -\alpha(x) \cdot v \cdot x^* \cdot x \cdot v^* \cdot \alpha(x)^* \end{aligned}$$

$$= -\alpha(x)\cdot v\cdot v^*\cdot\alpha(x^*)$$
$$= Q(v,v)\alpha(x)\cdot\alpha(x^*)$$
$$= Q(v,v)\alpha(x\cdot x^*) = Q(v,v).$$

We claim next that ρ is surjective. This follows from the standard fact (see Exercise 20.32) that the orthogonal group $\mathrm{O}(Q)$ is generated by reflections. Indeed, if R_w is the reflection in the hyperplane perpendicular to a vector w, normalized so that $Q(w, w) = -1$, it is easy to see that w is in $\mathrm{Pin}(Q)$ and $\rho(w) = R_w$; in fact,

$$w\cdot w^* = w\cdot(-w) = -Q(w,w) = 1,$$

and so

$$\rho(w)(w) = \alpha(w)\cdot w\cdot w^* = -w\cdot 1 = -w;$$

and if $Q(w, v) = 0$,

$$\rho(w)(v) = \alpha(w)\cdot v\cdot w^* = -w\cdot v\cdot w^* = v\cdot w\cdot w^* = v.$$

The next claim is that the kernel of ρ on the larger group $\mathrm{Pin}(Q)$ is ± 1. Suppose x is in the kernel, and write $x = x_0 + x_1$ with $x_0 \in C^{\text{even}}$ and $x_1 \in C^{\text{odd}}$. Then $x_0\cdot v = v\cdot x_0$ for all $v \in V$, so x_0 is in the center of C. And $x_1\cdot v = -v\cdot x_1$ for all $v \in V$. By Exercise 20.21, x_0 is in $\mathbb{C}\cdot 1$, and $x_1 = 0$. So $x = x_0$ is in $\mathbb{C}$ and $x^2 = 1$; so $x = \pm 1$.

It follows that if $R \in \mathrm{O}(Q)$ is written as a product of reflections $R_{w_1}\circ\ldots\circ R_{w_r}$, then the two elements in $\rho^{-1}(R)$ are $\pm w_1\cdot\ldots\cdot w_r$. In particular, we get another description of the spin groups:

$$\begin{aligned}\mathrm{Spin}(Q) &= \mathrm{Pin}(Q)\cap C(Q)^{\text{even}} = \rho^{-1}(\mathrm{SO}(Q))\\ &= \{\pm w_1\cdot\ldots\cdot w_{2k} \colon w_i \in V,\ Q(w_i, w_i) = -1\}.\end{aligned} \tag{20.31}$$

Since $-1 = v\cdot v$ for any v with $Q(v, v) = -1$, we see that the spin group consists of even products of such elements.

To complete the proof, we must check that $\mathrm{Spin}(Q)$ is connected or, equivalently, that the two elements in the kernel of ρ can be connected by a path. We leave this now as an exercise, since much more will be seen shortly. □

Exercise 20.32*. Let Q be a nondegenerate symmetric bilinear form on a real or complex vector space V.

(a) Show that if v and w are vectors in V with $Q(v, v) = Q(w, w) \neq 0$, then there is either a reflection or a product of two reflections that takes v into w.

(b) Deduce that every element of the orthogonal group of Q can be written as the product of at most $2\cdot\dim(V)$ reflections.

Exercise 20.33*. Since $\mathrm{Spin}(Q)$ is a subgroup of the multiplicative group of $C(Q)$, its Lie algebra is a subalgebra of $C(Q)$ with its usual bracket. Verify that this subalgebra is the subalgebra $\mathfrak{so}(Q)$ that was constructed in §20.1.

Exercise 20.34. The fact that $\bigwedge^{\bullet} W$ (and $\bigwedge^{\bullet} W'$ in the odd case) is an irreducible module over $C(Q)$ is equivalent to the fact that it is an irreducible representation of the group $\mathrm{Pin}(Q)$ since the linear span of $\mathrm{Pin}(Q)$ is dense in $C(Q)$.

(a) Apply the analysis of §5.1 to the subgroup

$$\mathrm{Spin}(Q) \subset \mathrm{Pin}(Q)$$

of index two. In the odd case, $\bigwedge^{\bullet} W$ and $\bigwedge^{\bullet} W'$ are conjugate representations, so their restrictions to $\mathrm{Spin}(Q)$ are isomorphic and irreducible: this is the spin representation. In the even case, $\bigwedge^{\bullet} W$ is self-conjugate, and its restriction to $\mathrm{Spin}(Q)$ is a sum of two conjugate irreducible representations, which are the two half-spin representations.

(b) Of the representations of $\mathrm{Spin}(Q)$ (i.e., the representations of $\mathfrak{so}_m\mathbb{C}$), which induce irreducible representations of $\mathrm{Pin}(Q)$ and which are restrictions of irreducible representations of $\mathrm{Pin}(Q)$?

Exercise 20.35. Deduce the irreducibility of the spin and half-spin representations from the fact that their restrictions to the 2-groups of Exercise 3.9 are irreducible representations of these finite groups.

Exercise 20.36*. Show that the center of $\mathrm{Spin}_m(\mathbb{C})$ is $\rho^{-1}(1) = \{\pm 1\}$ if m is odd. If m is even show that the center is

$$\rho^{-1}(\pm 1) = \{\pm 1, \pm\omega\},$$

where, in terms of our standard basis,

$$\omega = \frac{ie_1 \cdot e_{n+1} - ie_{n+1} \cdot e_1}{2} \cdot \ldots \cdot \frac{ie_n \cdot e_{2n} - ie_{2n} \cdot e_n}{2}.$$

Exercise 20.37*. Show that the spin representation $\mathrm{Spin}(Q) \to \mathrm{GL}(S)$ maps into the special linear group $\mathrm{SL}(S)$. Show that for $m = 2n$ and n even, the half-spin representations also map into the special linear groups $\mathrm{SL}(S^+)$ and $\mathrm{SL}(S^-)$.

Exercise 20.38*. Construct a nondegenerate bilinear pairing β on the spinor space $S = \bigwedge^{\bullet} W$ by choosing an isomorphism of $\bigwedge^n W$ with $\mathbb{C}$ and letting $\beta(s, t)$ be the image of $\tau(s) \wedge t \in \bigwedge^{\bullet} W$ by the projection to $\bigwedge^n W = \mathbb{C}$, where τ is the main antiautomorphism).

(a) When $m = 2n$, show that β can also be defined by the identity $\beta(s, t)f = \tau(s \cdot f) \cdot t \cdot f$ for an appropriate generator f of $\bigwedge^n W'$. Deduce that the action of $\mathrm{Spin}(Q)$ on S respects the bilinear form β.

(b) Show that β is symmetric if n is congruent to 0 or 3 modulo 4, and skew-symmetric otherwise. So the spin representation is a homomorphism

$$\mathrm{Spin}_{2n+1}\mathbb{C} \to \mathrm{SO}_{2n}\mathbb{C} \quad \text{if } n \equiv 0, 3\ (4),$$

$$\mathrm{Spin}_{2n+1}\mathbb{C} \to \mathrm{Sp}_{2n}\mathbb{C} \quad \text{if } n \equiv 1, 2\ (4).$$

(c) If $m = 2n$, the restrictions of β to S^+ and S^- are zero if n is odd. For n even, deduce that the half-spin representations are homomorphisms

$$\mathrm{Spin}_{2n}\mathbb{C} \to \mathrm{SO}_{2^{n-1}}\mathbb{C} \quad \text{if } n \equiv 0\ (4),$$

$$\mathrm{Spin}_{2n}\mathbb{C} \to \mathrm{Sp}_{2^{n-1}}\mathbb{C} \quad \text{if } n \equiv 2\ (4).$$

Note in particular that $\mathrm{Spin}_8\mathbb{C}$ has two maps to $\mathrm{SO}_8\mathbb{C}$ in addition to the original covering. "Triality," which we discuss in the next section, describes the relation among these three homomorphisms.

Exercise 20.39. Show that the spin and half-spin representations give the isomorphisms we have seen before:

$$\mathrm{Spin}_2\mathbb{C} \cong \mathrm{GL}(S^+) = \mathrm{GL}_1\mathbb{C} = \mathbb{C}^*,$$

$$\mathrm{Spin}_3\mathbb{C} \cong \mathrm{SL}(S) = \mathrm{SL}_2\mathbb{C},$$

$$\mathrm{Spin}_4\mathbb{C} \cong \mathrm{SL}(S^+) \times \mathrm{SL}(S^-) = \mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C},$$

$$\mathrm{Spin}_5\mathbb{C} \cong \mathrm{Sp}(S) = \mathrm{Sp}_4\mathbb{C},$$

$$\mathrm{Spin}_6\mathbb{C} \cong \mathrm{SL}(S^+) = \mathrm{SL}_4\mathbb{C}.$$

Exercise 20.40. Let C_m denote the Clifford algebra of the vector space $\mathbb{C}^m$ with our standard quadratic form Q_m.

(a) The embedding of $\mathbb{C}^{2n} = W \oplus W'$ in $\mathbb{C}^{2n+1} = W \oplus W' \oplus U$ as indicated induces an embedding of C_{2n} in C_{2n+1}, and corresponding embedding of $\mathrm{Spin}_{2n}\mathbb{C}$ in $\mathrm{Spin}_{2n+1}\mathbb{C}$ and of $\mathrm{SO}_{2n}\mathbb{C}$ in $\mathrm{SO}_{2n+1}\mathbb{C}$. Show that the spin representation S of $\mathrm{Spin}_{2n+1}\mathbb{C}$ restricts to the spin representation $S^+ \oplus S^-$ of $\mathrm{Spin}_{2n}\mathbb{C}$.

(b) Similarly there is an embedding of $\mathrm{Spin}_{2n+1}\mathbb{C}$ in $\mathrm{Spin}_{2n+2}\mathbb{C}$ coming from an embedding of $\mathbb{C}^{2n+1} = W \oplus W' \oplus U$ in $\mathbb{C}^{2n+2} = W \oplus W' \oplus U_1 \oplus U_2$; here $U_1 \oplus U_2 = \mathbb{C} \oplus \mathbb{C}$ with the quadratic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $U = \mathbb{C}$ is embedded in $U_1 \oplus U_2$ by sending 1 to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Show that each of the half-spin representations of $\mathrm{Spin}_{2n+2}\mathbb{C}$ restricts to the spin representation of $\mathrm{Spin}_{2n+1}\mathbb{C}$.

Very little of the above discussion needs to be changed to construct the real spin groups $\mathrm{Spin}_m(\mathbb{R})$, which are double coverings of the real orthogonal groups $\mathrm{SO}_m(\mathbb{R})$. One uses the real Clifford algebra $\mathrm{Cliff}(\mathbb{R}^m, Q)$ associated to the real quadratic form $Q = -Q_m$, where Q_m is the standard positive definite quadratic form on $\mathbb{R}^m$. If v_i are an orthonormal basis, the products in this Clifford algebra are given by

$$v_i \cdot v_j = -v_j \cdot v_i \quad \text{if } i \neq j, \qquad \text{and} \quad v_i \cdot v_i = -1.$$

The same definitions can be given as in the complex case, giving rise to coverings $\mathrm{Pin}_m(\mathbb{R})$ of $\mathrm{O}_m(\mathbb{R})$ and $\mathrm{Spin}_m(\mathbb{R})$ of $\mathrm{SO}_m(\mathbb{R})$.

Exercise 20.41. Show that $\mathrm{Spin}_m\mathbb{R}$ is connected by showing that if v and w are any two perpendicular elements in V with $Q(v, v) = Q(w, w) = -1$, the path

$$t \mapsto (\cos(t)v + \sin(t)w)\cdot(\cos(t)v - \sin(t)w), \quad 0 \leq t \leq \pi/2$$

connects -1 to 1.

Exercise 20.42. Show that $i \mapsto v_2 \cdot v_3$, $j \mapsto v_3 \cdot v_1$, $k \mapsto v_1 \cdot v_2$ determines an isomorphism of the quaternions $\mathbb{H}$ onto the even part of $\mathrm{Cliff}(\mathbb{R}^3, -Q_3)$, such that conjugation $^-$ in $\mathbb{H}$ corresponds to the conjugation $*$ in the Clifford algebra. Show that this maps $\mathrm{Sp}(2) = \{q \in \mathbb{H} \mid q\bar{q} = 1\}$ isomorphically onto $\mathrm{Spin}_3\mathbb{R}$, and that this isomorphism is compatible with the map to $\mathrm{SO}_3\mathbb{R}$ defined in Exercise 7.15.

More generally, if Q is a quadratic form on $\mathbb{R}^m$ with p positive and q negative eigenvalues, we get a group $\mathrm{Spin}^+(p, q)$ in the Clifford algebra $C(p, q) = \mathrm{Cliff}(\mathbb{R}^m, Q)$, with double coverings

$$\mathrm{Spin}^+(p, q) \to \mathrm{SO}^+(p, q).$$

Exercise 20.43*. Show that $\mathrm{Spin}^+(p, q)$ is connected if p and q are positive, except for the case $p = q = 1$, when it has two components. Show that if in the definition of spin groups one relaxes the condition $x \cdot x^* = 1$ to the condition $x \cdot x^* = \pm 1$, one gets coverings $\mathrm{Spin}(p, q)$ of $\mathrm{SO}(p, q)$.

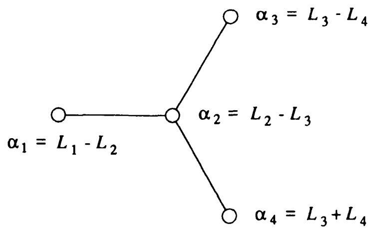
§20.3. $\mathrm{Spin}_8\mathbb{C}$ and Triality

When m is even, there is always an outer automorphism of $\mathrm{Spin}_m(\mathbb{C})$ that interchanges the two spin representations S^+ and S^-, while preserving the basic representation $V = \mathbb{C}^m$ (cf. Exercise 19.9). In case $m = 8$, all three of these representations V, S^+, and S^- are eight dimensional. One basic expression of *triality* is the fact that there are automorphisms of $\mathrm{Spin}_8\mathbb{C}$ or $\mathfrak{so}_8\mathbb{C}$ that permute these three representations arbitrarily. (In fact, the group of outer automorphisms modulo inner automorphisms is the symmetric group on three elements.) We give a brief discussion of this phenomenon in this section, in the form of an extended exercise.

To see where these automorphisms might come from, consider the four simple roots:

$$\alpha_1 = L_1 - L_2, \quad \alpha_2 = L_2 - L_3, \quad \alpha_3 = L_3 - L_4, \quad \alpha_4 = L_3 + L_4.$$

Note that α_1, α_3, and α_4 are mutually perpendicular, and that each makes an angle of 120° with α_2:



Exercise 20.44*. For each of the six permutations of $\{\alpha_1, \alpha_3, \alpha_4\}$ find the orthogonal automorphism of the root space which fixes α_2 and realizes the permutation of α_1, α_3, and α_4.

Each automorphism of this exercise corresponds to an automorphism of the Cartan subalgebra $\mathfrak{h}$. In the next lecture we will see that such automorphisms can be extended (nonuniquely) to automorphisms of the Lie algebra $\mathfrak{so}_8(\mathbb{C})$. (For explicit formulas see [Ca2].)

There is also a purely geometric notion of triality. Recall that an even-dimensional quadric Q can contain linear spaces Λ of at most half the dimension of Q, and that there are two families of linear spaces of this maximal dimension (cf. [G-H], [Ha]). In case Q is six-dimensional, each of these families can themselves be realized as six-dimensional quadrics, which we may denote by Q^+ and Q^- (see below). Moreover, there are correspondences that assign to a point of any one of these quadrics a 3-plane in each of the others:

$$\begin{tikzcd}
& \text{Point in } Q \arrow[r] \arrow[dl] & \text{3-plane in } Q^+ & \\
\text{3-plane in } Q^- & & & \text{Point in } Q^- \arrow[ul] \arrow[dl] \\
& \text{Point in } Q^+ \arrow[r] \arrow[ul] & \text{3-plane in } Q &
\end{tikzcd} \tag{20.45}$$

Given $P \in Q$, $\{\Lambda \in Q^+ : \Lambda \text{ contains } P\}$ is a 3-plane in Q^+, and $\{\Lambda \in Q^- : \Lambda$ contains $P\}$ is a 3-plane in Q^-.

Given $\Lambda \in Q^+$, Λ itself is a 3-plane in Q, and $\{\Gamma \in Q^- : \Gamma \cap \Lambda \text{ is a 2-plane}\}$ is a 3-plane in Q^-.

Given $\Lambda \in Q^-$, Λ itself is a 3-plane in Q, and $\{\Gamma \in Q^+ : \Gamma \cap \Lambda \text{ is a 2-plane}\}$ is a 3-plane in Q^+.

To relate these two notions of triality, take Q to be our standard quadric in $\mathbb{P}^7 = \mathbb{P}(V)$, with $V = W \oplus W'$ with our usual quadratic space, and let $S^+ = \wedge^{\text{even}} W$ and $S^- = \wedge^{\text{odd}} W$ be the two spin representations. In Exercise 20.38 we constructed quadratic forms on S^+ and S^-, by choosing an isomorphism of $\wedge^4 W$ with $\mathbb{C}$. This gives us two quadrics Q^+ and Q^- in $\mathbb{P}(S^+)$ and $\mathbb{P}(S^-)$.

To identify Q^+ and Q^- with the families of 3-planes in Q, recall the action of V on $S = \wedge^\bullet W = S^+ \oplus S^-$ which gave rise to the isomorphism of the Clifford algebra with $\mathrm{End}(S)$ (cf. Lemma 20.9). This in fact maps S^+ to S^-

and S^- to S^+; so we have bilinear maps

$$V \times S^+ \dashrightarrow S^- \quad \text{and} \quad V \times S^- \dashrightarrow S^+. \tag{20.46}$$

Exercise 20.47. Show that for each point in Q^+, represented by a vector $s \in S^+$, $\{v \in V: v \cdot s = 0\}$ is an isotropic 4-plane in V, and hence determines a projective 3-plane in Q. Similarly, each point in Q^- determines a 3-plane in Q. Show that every 3-plane in Q arises uniquely in one of these ways.

Let $\langle\ ,\ \rangle_V$ denote the symmetric form corresponding to the quadratic form in V, and similarly for S^+ and S^-. Define a product

$$S^+ \times S^- \dashrightarrow V, \qquad s \times t \mapsto s \cdot t, \tag{20.48}$$

by requiring that $\langle v, s \cdot t\rangle_V = \langle v \cdot s, t\rangle_{S^-}$ for all $v \in V$.

Exercise 20.49. Use this product, together with those in (20.46), to show that the other four arrows in the hexagon (20.45) for geometric triality can be described as in the preceding exercise.

This leads to an algebraic version of triality, which we sketch following [Ch2]. The above products determine a commutative but nonassociative product on the direct sum $A = V \oplus S^+ \oplus S^-$. The operation

$$(v, s, t) \mapsto \langle v \cdot s, t\rangle_{S^-}$$

determines a cubic form on A, which by polarization determines a symmetric trilinear form Φ on A.

Exercise 20.50*. One can construct an automorphism J of A of order three that sends V to S^+, S^+ to S^-, and S^- to V, preserving their quadratic forms, and compatible with the cubic form. The definition of J depends on the choice of an element $v_1 \in V$ and $s_1 \in S^+$ with $\langle v_1, v_1\rangle_V = \langle s_1, s_1\rangle_{S^+} = 1$; set $t_1 = v_1 \cdot s_1$, so that $\langle t_1, t_1\rangle_{S^-} = 1$ as well. The map J is defined to be the composite $\mu \circ \nu$ of two involutions μ and ν, which are determined by the following:

(i) μ interchanges S^+ and S^-, and maps V to itself, with $\mu(s) = v_1 \cdot s$ for $s \in S^+$; $\mu(v) = 2\langle v, v_1\rangle_V v_1 - v$ for $v \in V$.
(ii) ν interchanges V and S^-, maps S^+ to itself, with $\nu(v) = v \cdot s_1$ for $v \in V$; $\nu(s) = 2\langle s, s_1\rangle_{S^+} s_1 - s$ for $s \in S^+$.

Show that this J satisfies the asserted properties.

Exercise 20.51*. In this algebraic form, triality can be expressed by the assertion that there is an automorphism j of $\mathrm{Spin}_8\mathbb{C}$ of order 3 compatible with J, i.e., such that for all $x \in \mathrm{Spin}_8\mathbb{C}$, the following diagrams commute:

$$\begin{tikzcd}
V \arrow[r, "J"] \arrow[d, "\rho(x)"] & S^+ \arrow[r, "J"] \arrow[d, "\rho^+(j(x))"] & S^- \arrow[r, "J"] \arrow[d, "\rho^-(j^2(x))"] & V \arrow[d, "\rho(x)"] \\
V \arrow[r, "J"'] & S^+ \arrow[r, "J"'] & S^- \arrow[r, "J"'] & V
\end{tikzcd}$$

If $j' \colon \mathfrak{so}_8\mathbb{C} \to \mathfrak{so}_8\mathbb{C}$ is the map induced by j, the fact that j is compatible with the trilinear form Φ (cf. Exercise 20.49) translates to the "local triality" equation

$$\Phi(Xv, s, t) + \Phi(v, Ys, t) + \Phi(v, s, Zt) = 0$$

for $X \in \mathfrak{so}_8\mathbb{C}$, $Y = j'(X)$, $Z = j'(Y)$.

PART IV
LIE THEORY

The purpose of this final part of the book is threefold.

First of all, we want to complete the program stated in the introduction to Part II. We have completed the first two steps of this program, showing in Part II how the analysis of representations of Lie groups could be reduced to the study of representations of complex Lie algebras, of which the most important are the semisimple; and carrying out in Part III such an analysis for the classical Lie algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_m\mathbb{C}$. To finish the story, we want now to translate our answers back into the terms of the original problem. In particular, we want to deal with representations of Lie groups as well as Lie algebras, and real groups and algebras as well as complex. The passage back to groups is described in Lecture 21, and the analysis of the real case in Lecture 26.

Another goal of this Part is to establish a framework for some of the results of the preceding lectures—to describe the general theory of semisimple Lie algebras and Lie groups. The key point here is the introduction of the Dynkin diagram and its use in classifying all semisimple Lie algebras over $\mathbb{C}$. From one point of view, the impact of the classification theorem is not great: it just tells us that we have in fact already analyzed all but five of the simple Lie algebras in existence. Beyond that, however, it provides a picture and a language for the description of the general Lie algebra. This both yields a description of the five remaining simple Lie algebras and allows us to give uniform descriptions of associated objects: for example, the compact homogeneous spaces associated to simple Lie groups, or the characters of their representations. The classification theory of semisimple Lie algebras is given in Lecture 21; the description in these terms of their representations and characters is given in Lecture 23. The five exceptional simple Lie algebras, whose existence is revealed from the Dynkin diagrams, are studied in Lecture

22; we give a fairly detailed account of one of them ($\mathfrak{g}_2$), with only brief descriptions of the others.

Third, all this general theory makes it possible to answer the main outstanding problem left over from Part III: a description of the multiplicities of the weights in the irreducible representations of the simple Lie algebras. We give in Lectures 24 and 25 a number of formulas for these multiplicities.

This, it should be said, represents in some ways a shift in style. In the previous lectures we would typically analyze special cases first and deduce general patterns from these cases; here, for example, the Weyl character formula is stated and proved in general, then specialized to the various individual cases (this is the approach more often taken in the literature on the subject). In some ways, this is a fourth goal of Part IV: to provide a bridge between the naive exploration of Lie theory undertaken in Parts II and III, and the more general theory readers will find elsewhere when they pursue the subject further.

Finally, we should repeat here the disclaimer made in the Preface. This part of the book, to the extent that it is successful, will introduce the reader to the rich and varied world of Lie theory; but it certainly undertakes no serious exploration of that world. We do not, for example, touch on such basic constructions as the universal enveloping algebra, Verma modules, Tits buildings; and we do not even hint at the fascinating subject of (infinite-dimensional) unitary representations. The reader is encouraged to sample these and other topics, as well as those included here, according to background and interest.

LECTURE 21

The Classification of Complex Simple Lie Algebras

In the first section of this lecture we introduce the Dynkin diagram associated to a semisimple Lie algebra $\mathfrak{g}$. This is an amazingly efficient way of conveying the structure of $\mathfrak{g}$: it is a simple diagram that not only determines $\mathfrak{g}$ up to isomorphism in theory, but in practice exhibits many of the properties of $\mathfrak{g}$. The main use of Dynkin diagrams in this lecture, however, will be to provide a framework for the basic classification theorem, which says that with exactly five exceptions the Lie algebras discussed so far in these lectures are all the simple Lie algebras. To do this, in §21.2 we show how to list all diagrams that arise from semisimple Lie algebras. In §21.3 we show how to recover such a Lie algebra from the data of its diagram, completing the proof of the classification theorem. All three sections are completely elementary, though §21.3 gets a little complicated; it may be useful to read it in conjunction with §22.1, where the process described is carried out in detail for the exceptional algebra $\mathfrak{g}_2$. (Note that neither §21.3 or §22.1 is a prerequisite for §22.3, where another description of $\mathfrak{g}_2$ will be given.)

§21.1: Dynkin diagrams associated to semisimple Lie algebras
§21.2: Classifying Dynkin diagrams
§21.2: Recovering a Lie algebra from its Dynkin diagram

§21.1. Dynkin Diagrams Associated to Semisimple Lie Algebras

For the following, we will let $\mathfrak{g}$ be a semisimple Lie algebra; as usual, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ will be fixed throughout. As we have seen, the roots R of $\mathfrak{g}$ span a real subspace of $\mathfrak{h}^*$ on which the Killing form is positive definite. We denote this Euclidean space here by $\mathbb{E}$, and the Killing form on $\mathbb{E}$ simply by

(,) instead of $B($, $)$. The geometry of how R sits in $\mathbb{E}$ is very rigid, as indicated by the pictures we have seen for the classical Lie algebras. In this section we will classify the possible configurations, up to rotation and multiplication by a positive scalar in $\mathbb{E}$. In the next section we will see that this geometry completely determines the Lie algebra.

The following four properties of the root system are all that are needed:

(1) *R is a finite set spanning $\mathbb{E}$.*
(2) *$\alpha \in R \Rightarrow -\alpha \in R$, but $k \cdot \alpha$ is not in R if k is any real number other than ± 1.*
(3) *For $\alpha \in R$, the reflection W_α in the hyperplane $\alpha^\perp$ maps R to itself.*
(4) *For $\alpha, \beta \in R$, the real number*

$$n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

is an integer.

Except perhaps for the second part of (2), these properties have been seen in Lecture 14. For example, (4) is Corollary 14.29. Note that $n_{\beta\alpha} = \beta(H_\alpha)$, and

$$W_\alpha(\beta) = \beta - n_{\beta\alpha}\alpha. \tag{21.1}$$

For (2), consider the representation $\mathfrak{i} = \bigoplus_k \mathfrak{g}_{k\alpha}$ of the Lie algebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$. Note that all the nonzero factors but $\mathfrak{h} = \mathfrak{g}_0$ are one dimensional. We may assume α is the smallest nonzero root that appears in the string. Now, decompose $\mathfrak{i}$ as an $\mathfrak{s}_\alpha$-module:

$$\mathfrak{i} = \mathfrak{s}_\alpha \oplus \mathfrak{i}'.$$

By the hypothesis that α is the smallest nonzero root that appears in the string, $\mathfrak{i}'$ is a representation of $\mathfrak{s}_\alpha$ having no eigenspace with eigenvalue 1 or 2 for H_α. It follows that $\mathfrak{i}'$ must be trivial, i.e., $\mathfrak{g}_{k\alpha} = (0)$ for $k \neq 0$ or ± 1.

Any set R of elements in a Euclidean space $\mathbb{E}$ satisfying conditions (1) to (4) may be called an (*abstract*) *root system*.

Property (4) puts very strong restrictions on the geometry of the roots. If ϑ is the angle between α and β, we have

$$n_{\beta\alpha} = 2\cos(\vartheta)\frac{\|\beta\|}{\|\alpha\|}. \tag{21.2}$$

In particular,

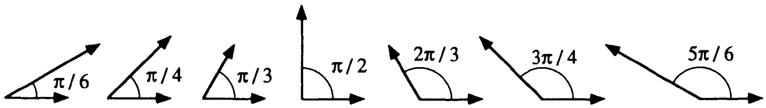
$$n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2(\vartheta) \tag{21.3}$$

is an integer between 0 and 4. The case when this integer is 4 occurs when $\cos(\vartheta) = \pm 1$, i.e. $\beta = \pm\alpha$. Omitting this trivial case, the only possibilities are therefore those given in the following table. Here we have ordered the two roots so that $\|\beta\| \geq \|\alpha\|$, or $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$.

Table 21.4

$\cos(\vartheta)$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
ϑ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$n_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$n_{\alpha\beta}$	1	1	1	0	-1	-1	-1
$\dfrac{\|\beta\|}{\|\alpha\|}$	$\sqrt{3}$	$\sqrt{2}$	1	$*$	1	$\sqrt{2}$	$\sqrt{3}$

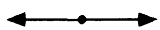
In other words, the relation of any two roots α and β is one of



The dimension $n = \dim_{\mathbb{R}}\mathbb{E} = \dim_{\mathbb{C}}\mathfrak{h}$ is called the *rank* (of the Lie algebra, or the root system). It is easy to find all those of smallest ranks. As we write them down, we will label them by the labels (A_n), (B_n), ... that have become standard.

Rank 1. The only possibility is

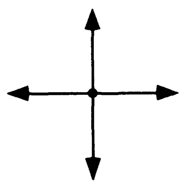
(A_1)



which is the root system of $\mathfrak{sl}_2\mathbb{C}$.

Rank 2. Note first that by Property (3), the angle between two roots must be the same for any pair of adjacent roots in a two-dimensional root system. As we will see, any of the four angles $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$ can occur; once this angle is specified the relative lengths of the roots are determined by Property (4), except in the case of right angles. Thus, up to scalars there are exactly four root systems of dimension two. First we have the case $\vartheta = \pi/2$,

$(A_1 \times A_1)$



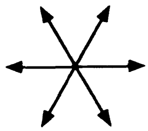
which is the root system of $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_4\mathbb{C}$.

(In general, the orthogonal direct sum of two root systems is a root system;

a root system that is not such a sum is called *irreducible*. Our task will be to classify all irreducible root systems.)

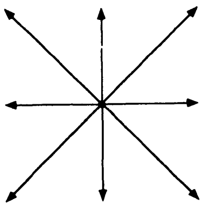
The other root systems of rank 2 are

(A_2)



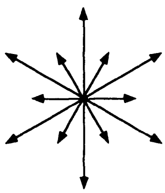
the root system of $\mathfrak{sl}_3\mathbb{C}$;

(B_2)



the root system of $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$; and

(G_2)



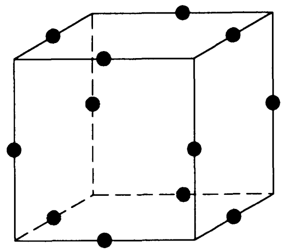
Although we have not yet seen a Lie algebra with this root system, we will see that there is one.

Exercise 21.5. Show that these are all the root systems of rank 2.

Exercise 21.6. Show that a semisimple Lie algebra is simple if and only if its root system is irreducible.

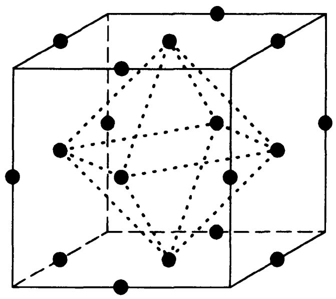
Rank 3. Besides the direct sums of (A_1) with one of those of rank 2, we have the irreducible root systems we have seen; we draw only dots at the ends of the vectors, the origins being in the centers of the reference cubes:

(A_3)



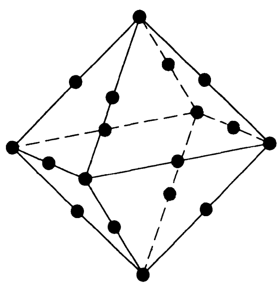
which is the root system of $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$;

(B_3)



the root system of $\mathfrak{so}_7\mathbb{C}$;

(C_3)



the root system of $\mathfrak{sp}_6\mathbb{C}$.

Exercise 21.7. Show that there are no other root systems of rank 3.

We can further reduce the data of a root system by introducing a subset of the roots, called the simple roots. First, choose as in Lecture 14 a direction

$l\colon \mathbb{E} \to \mathbb{R}$, so that $R = R^+ \cup R^-$ is a disjoint union of positive and negative roots. Call a positive root *simple* if it is not the sum of two other positive roots. For the classical Lie algebras, keeping the notations and conventions of Lectures 15–20, the simple roots are

(A_n)	$\mathfrak{sl}_{n+1}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n - L_{n+1}$,
(B_n)	$\mathfrak{so}_{2n+1}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n$,
(C_n)	$\mathfrak{sp}_{2n}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, 2L_n$,
(D_n)	$\mathfrak{so}_{2n}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_{n-1} + L_n$.

Exercise 21.8. Verify this list, and find two simple roots for (G_2).

We next deduce a few consequences of properties (1)–(4), which indicate how strong these axioms are. They will be used in the present classification of abstract systems, as well as in the following section.

(5) *If α, β are roots with $\beta \neq \pm\alpha$, then the α-string through β, i.e., the roots of the form*

$$\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + q\alpha$$

has at most four in a string, i.e. $p + q \leq 3$; in addition, $p - q = n_{\beta\alpha}$.

Indeed, since $W_\alpha(\beta + q\alpha) = \beta - p\alpha$, and

$$W_\alpha(\beta + q\alpha) = (\beta - n_{\beta\alpha}\alpha) - q\alpha,$$

we must have $p = n_{\beta\alpha} + q$, which is the second equality. For the first, we may take $p = 0$, and then $q = -n_{\beta\alpha}$, which we have seen is an integer no larger than three. As a consequence of (5) we have

(6) *Suppose α, β are roots with $\beta \neq \pm\alpha$. Then*

$(\beta, \alpha) > 0 \Rightarrow \alpha - \beta$ is a root;
$(\beta, \alpha) < 0 \Rightarrow \alpha + \beta$ is a root.

If $(\beta, \alpha) = 0$, then $\alpha - \beta$ and $\alpha + \beta$ are simultaneously roots or nonroots.

(7) *If α and β are distinct simple roots, then $\alpha - \beta$ and $\beta - \alpha$ are not roots.*

This follows from the definition of simple, since from the equation $\alpha = \beta + (\alpha - \beta)$, $\alpha - \beta$ cannot be in R^+, and similarly $-(\alpha - \beta) = \beta - \alpha$ cannot be in R^+. From (6) and (7) we deduce that $(\alpha, \beta) \leq 0$, i.e.,

(8) *The angle between two distinct simple roots cannot be acute.*

(9) *The simple roots are linearly independent.*

This follows from (8) by

Exercise 21.9*. If a set of vectors lies on one side of a hyperplane, with all mutual angles at least 90°, show that they must be linearly independent.

(10) *There are precisely n simple roots. Each positive root can be written uniquely as a non-negative integral linear combination of simple roots.*

Since R spans $\mathbb{E}$, the first statement follows from (9), as does the uniqueness of the second statement. The fact that any positive root can be written as a positive sum of simple roots follows readily from the definition, for if α were a positive root with minimal $l(\alpha)$ that could not be so written, then α is not simple, so $\alpha = \beta + \gamma$, with β and γ positive roots with $l(\beta)$, $l(\gamma) < l(\alpha)$.

Note that as an immediate corollary of (10) it follows that *no root is a linear combination of the simple roots α_i with coefficients of mixed sign.* For example, (7) is just a special case of this.

The *Dynkin diagram* of the root system is drawn by drawing one node $\bigcirc$ for each simple root and joining two nodes by a number of lines depending on the angle ϑ between them:

no lines	$\bigcirc \quad \bigcirc$	if $\vartheta = \pi/2$
one line	$\bigcirc\!\!-\!\!\!-\!\!\bigcirc$	if $\vartheta = 2\pi/3$
two lines	$\bigcirc\!\Rightarrow\!\bigcirc$	if $\vartheta = 3\pi/4$
three lines	$\bigcirc\!\Rrightarrow\!\bigcirc$	if $\vartheta = 5\pi/6$.

When there is one line, the roots have the same length; if two or three lines, an arrow is drawn pointing from the *longer* to the *shorter* root.

Exercise 21.10. Show that a root system is irreducible if and only if its Dynkin diagram is connected.

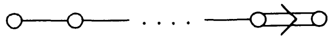
We will see later that the Dynkin diagram of a root system is independent of the choice of direction, i.e., of the decomposition of R into R^+ and R^-.

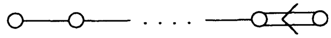
§21.2. Classifying Dynkin Diagrams

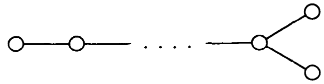
The wonderful thing about Dynkin diagrams is that from this very simple picture one can reconstruct the entire Lie algebra from which it came. We will see this in the following section; for now, we ask the complementary question of which diagrams arise from Lie algebras. Our goal is the following classification theorem, which is a result in pure Euclidean geometry. (The subscripts on the labels (A_n), ... are the number of nodes.)

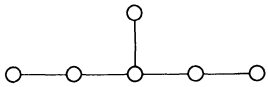
Theorem 21.11. *The Dynkin diagrams of irreducible root systems are precisely*:

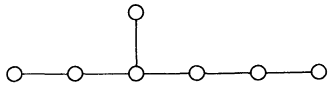
(A_n)  ($n \geq 1$)

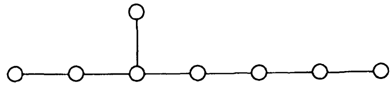
(B_n)  ($n \geq 2$)

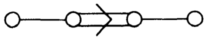
(C_n)  ($n \geq 3$)

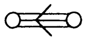
(D_n)  ($n \geq 4$)

(E_6) 

(E_7) 

(E_8) 

(F_4) 

(G_2) 

The first four are those belonging to the classical series we have been studying:

(A_n)	$\mathfrak{sl}_{n+1}\mathbb{C}$
(B_n)	$\mathfrak{so}_{2n+1}\mathbb{C}$
(C_n)	$\mathfrak{sp}_{2n}\mathbb{C}$
(D_n)	$\mathfrak{so}_{2n}\mathbb{C}$

The restrictions on n in these series are to avoid repeats, as well as degenerate cases. Indeed, the diagrams can be used to recall all the coincidences we have seen:

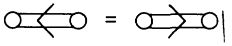
When $n = 1$, all four of the diagrams become one node. The case (D_1) is degenerate, since $\mathfrak{so}_2\mathbb{C}$ is not semisimple, while the coincidences (C_1) = (B_1) = (A_1) correspond to the isomorphisms

$$\mathfrak{sp}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \qquad \circ.$$

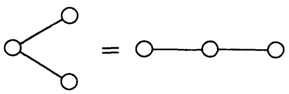
For $n = 2$, $(D_2) = (A_1) \times (A_1)$ consists of two disjoint nodes, corresponding to the isomorphism

$\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$  .

The coincidence $(C_2) = (B_2)$ corresponds to the isomorphism

$\mathfrak{sp}_4\mathbb{C} \cong \mathfrak{so}_5\mathbb{C}$  .

For $n = 3$, the fact that $(D_3) = (A_3)$ reflects the isomorphism

$\mathfrak{so}_6\mathbb{C} \cong \mathfrak{sl}_4\mathbb{C}$  .

Proof of the Theorem. Our desert-island reader would find this a pleasant pastime. For example, if there are two simple roots with angle $5\pi/6$, the plane of these roots must contain the G_2 configuration of 12 roots. It is not hard to see that one cannot add another root that is not perpendicular to this plane, without some of the 12 angles and lengths being wrong. This shows that (G_2) is the only connected diagram containing a triple line. At the risk of spoiling your fun, we give the general proof of a slightly stronger result.

In fact, the angles alone determine the possible diagrams. Such diagrams, without the arrows to indicate relative lengths, are often called *Coxeter diagrams* (or Coxeter graphs). Define a diagram of n nodes, with each pair connected by 0, 1, 2, or 3 lines, to be *admissible* if there are n independent unit vectors $e_1, \ldots, e_n$ in a Euclidean space $\mathbb{E}$ with the angle between e_i and e_j being $\pi/2$, $2\pi/3$, $3\pi/4$, or $5\pi/6$, according as the number of lines between corresponding nodes is 0, 1, 2, or 3. The claim is that the diagrams of the above Dynkin diagrams, ignoring the arrows, are the only connected admissible diagrams. Note that

$$(e_i, e_j) = 0, \ -1/2, \ -\sqrt{2}/2, \text{ or } -\sqrt{3}/2, \tag{21.12}$$

according as the number of lines between them is 0, 1, 2, or 3; equivalently,

$$4(e_i, e_j)^2 = \text{number of lines between } e_i \text{ and } e_j. \tag{21.13}$$

The steps of the proof are as follows:

(i) *Any subdiagram of an admissible diagram, obtained by removing some nodes and all lines to them, will also be admissible.*

(ii) *There are at most $n - 1$ pairs of nodes that are connected by lines. The diagram has no cycles (loops).*

Indeed, if e_i and e_j are connected, $2(e_i, e_j) \leq -1$, and

$$0 < \left(\sum e_i, \sum e_i\right) = n + 2 \sum_{i<j} (e_i, e_j),$$

which proves the first statement of (ii). The second follows from the first and (i).

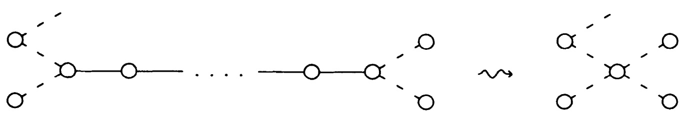
(iii) *No node has more than three lines to it.*

By (i), we may assume that e_1 is connected to each of the other nodes; by (ii), no other nodes are connected to each other. We must show that $\sum_{j=2}^{n} 4(e_1, e_j)^2 < 4$. Since $e_2, \ldots, e_n$ are perpendicular unit vectors, and e_1 is not in their span,

$$1 = (e_1, e_1)^2 > \sum_{j=2}^{n} (e_1, e_j)^2,$$

as required.

(iv) *In an admissible diagram, any string of nodes connected to each other by one line, with none but the ends of the string connected to any other nodes, can be collapsed to one node, and resulting diagram remains admissible*:

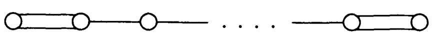


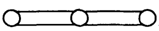
If $e_1, \ldots, e_r$ are the unit vectors corresponding to the string of nodes, then $e' = e_1 + \cdots + e_r$ is a unit vector, since

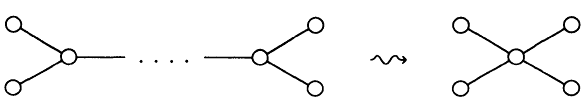
$$\begin{aligned}(e', e') &= r + 2((e_1, e_2) + (e_2, e_3) + \cdots + (e_{r-1}, e_r)) \\ &= r - (r - 1).\end{aligned}$$

Moreover, e' satisfies the same conditions with respect to the other vectors since (e', e_j) is either (e_1, e_j) or (e_r, e_j).

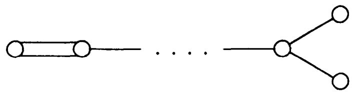
Now we can rule out the other admissible connected diagrams not on our list. First, from (iii) we see that the diagram (G_2) has the only triple edge. Next, there cannot be two double lines, or we could find a subdiagram of the form:



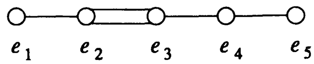
and then collapse the middle to get , contradicting (iii). Similarly there can be at most one triple node, i.e., a node with single lines to three other nodes, by



By the same reasoning, there cannot be a triple node together with a double line:



To finish the case with double lines, we must simply verify that



is not admissible. Consider general vectors $v = a_1e_1 + a_2e_2$, and $w = a_3e_3 + a_4e_4 + a_5e_5$. We have

$$\|v\|^2 = a_1^2 + a_2^2 - a_1a_2, \qquad \|w\|^2 = a_3^2 + a_4^2 + a_5^2 - a_3a_4 - a_4a_5,$$

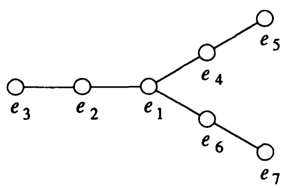
and $(v, w) = -a_2a_3/\sqrt{2}$. We want to choose v and w to contradict the Cauchy–Schwarz inequality $(v, w)^2 < \|v\|^2\|w\|^2$. For this we want $|a_2|/\|v\|$ and $|a_3|/\|w\|$ to be as large as possible.

Exercise 21.14. Show that these maxima are achieved by taking $a_2 = 2a_1$ and $a_3 = 3a_5$, $a_4 = 2a_5$.

In fact, $v = e_1 + 2e_2$, $w = 3e_3 + 2e_4 + e_5$ do give the contradictory

$$(v, w)^2 = 18, \quad \|v\|^2 = 3, \quad \text{and} \quad \|w\|^2 = 6.$$

Finally, we must show that the strings coming out from a triple node cannot be longer that those specified in types (D_n), (E_6), (E_7), or (E_8). First, we rule out



Consider the three perpendicular unit vectors:

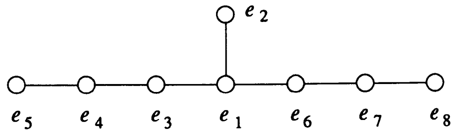
$$u = (2e_2 + e_3)/\sqrt{3}, \qquad v = (2e_4 + e_5)/\sqrt{3}, \qquad w = (2e_6 + e_7)/\sqrt{3}.$$

Then as in (iii), since e_1 is not in the span of them,

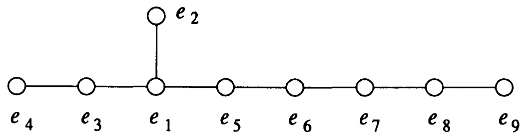
$$1 = \|e_1\|^2 > (e_1, u)^2 + (e_1, v)^2 + (e_1, w)^2 = 1/3 + 1/3 + 1/3 = 1,$$

a contradiction.

Exercise 21.15*. Similarly, rule out



and



(The last few arguments can be amalgamated, by showing that if the legs from a triple node have lengths p, q, and r, then $1/p + 1/q + 1/r$ must be greater than 1.)

This finishes the proof of the theorem. □

§21.3. Recovering a Lie Algebra from Its Dynkin Diagram

In this section we will complete the classification theorem for simple Lie algebras by showing how one may recover a simple Lie algebra from the data of its Dynkin diagram. This will proceed in two stages: first, we will see how to reconstruct a root system from its Dynkin diagram (which a priori only tells us the configuration of the simple roots). Secondly, we will show how to describe the entire Lie algebra in terms of its root system. (In the next lecture we will do all this explicitly, by hand, and independently of the general discussion here, for the simplest exceptional case (G_2); as we have noted, the reader may find it useful to work through §22.1 before or while reading the general story described here.)

To begin with, to recover the root system from the Dynkin diagram, let $\alpha_1, \ldots, \alpha_n$ be the simple roots corresponding to the nodes of a connected Dynkin diagram. We must show which non-negative integral linear combinations $\sum m_i\alpha_i$ are roots. Call $\sum m_i$ the *level* of $\sum m_i\alpha_i$. Those of level one are the simple roots. For level two, we see from Property (2) that no $2\alpha_i$ is a root, and by

Property (6) that $\alpha_i + \alpha_j$ is a root precisely when $(\alpha_i, \alpha_j) < 0$, i.e., when the corresponding nodes are joined by a line.

Suppose we know all positive roots of level at most m, and let $\beta = \sum m_i \alpha_i$ be any positive root of level m. We next determine for each simple root $\alpha = \alpha_j$, whether $\beta + \alpha$ is also a root. Look at the α-string through β:

$$\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha.$$

We know p by induction (no root is a linear combination of the simple roots α_i with coefficients of mixed sign, so $p \leq m_j$ and $\beta - p\alpha$ is a positive root). By Property (5), $q = p - n_{\beta\alpha}$. So $\beta + \alpha$ is a root exactly when

$$p > n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = \sum_{i=1}^{n} m_i n_{\alpha_i \alpha}.$$

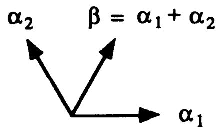
In effect, the additional roots we will find in this way are those obtained by reflecting a known positive root in the hyperplane perpendicular to a simple root α_i (and filling in the string if necessary).

To finish the proof, we must show that we get all the positive roots in this way. This will follow once from the fact that any positive root of level $m + 1$ can be written in at least one way as a sum of a positive root of level m and a simple root. If $\gamma = \sum r_i \alpha_i$ has level $m + 1$, from

$$0 < (\gamma, \gamma) = \sum r_i(\gamma, \alpha_i),$$

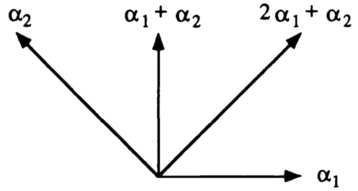
some (γ, α_i) must be positive, with $r_i > 0$. By property (6), $\gamma - \alpha_i$ is a root, as required.

By way of example, consider the rank 2 root systems. In the case of $\mathfrak{sl}_3\mathbb{C}$, we start with a pair of simple roots α_1, α_2 with $n_{\alpha_1, \alpha_2} = -1$, i.e., at an angle of $2\pi/3$; as always, we know that $\beta = \alpha_1 + \alpha_2$ is a root as well.



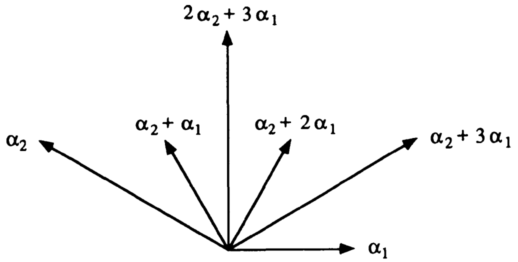
On the other hand, since $\beta - 2\alpha_1 = \alpha_2 - \alpha_1$ is not a root, $\beta + \alpha_1$ cannot be either, and likewise $\beta + \alpha_2$ is not; so we have all the positive roots.

In the case of $\mathfrak{sp}_4\mathbb{C}$, we have two simple roots α_1 and α_2 at an angle of $3\pi/4$; in terms of an orthonormal basis L_1 and L_2 these may be taken to be L_1 and $L_2 - L_1$, respectively.



We then see that in addition to $\beta = \alpha_1 + \alpha_2$, the sum $\beta + \alpha_1 = 2\alpha_1 + \alpha_2$ is a root—it is just the reflection of α_2 in the plane perpendicular to α_1—but $\beta + \alpha_2 = \alpha_1 + 2\alpha_2$ and $3\alpha_1 + \alpha_2$ are not because $\alpha_1 - \alpha_2$ and $\alpha_2 - \alpha_1$ are not respectively (alternatively, we could note that they would form inadmissible angles with α_1 and α_2 respectively).

Finally, in the case of (G_2), we have two simple roots α_1, α_2 at an angle of $5\pi/6$, which in terms of an orthonormal basis for $\mathbb{E}$ may be taken to be L_1 and $(-3L_1 + \sqrt{3}L_2)/2$ respectively.



Reflecting α_2 in the plane perpendicular to α_1 yields a string of roots $\alpha_2 + \alpha_1$, $\alpha_2 + 2\alpha_1$ and $\alpha_2 + 3\alpha_2$. Moreover, reflecting the last of these in the plane perpendicular to α_2 yields one more root, $2\alpha_2 + 3\alpha_3$. Finally, these are all the positive roots, giving us the root system for the diagram (G_2).

We state here the results of applying this process to the exceptional diagrams (F_4), (E_6), (E_7), and (E_8) (in addition to (G_2)). In each case, $L_1, \ldots, L_n$ is an orthogonal basis for $\mathbb{E}$, the simple roots α_i can be taken to be as follows, and the corresponding root systems are given:

(G_2)
$$\alpha_1 = L_1, \qquad \alpha_2 = -\frac{3}{2}L_1 + \frac{\sqrt{3}}{2}L_2;$$

$$R^+ = \left\{L_1, \sqrt{3}L_2, \pm L_1 + \frac{\sqrt{3}}{2}L_2, \pm\frac{3}{2}L_1 + \frac{\sqrt{3}}{2}L_2\right\}.$$

(G_2) thus has 6 positive roots.

(F_4)
$$\alpha_1 = L_2 - L_3, \qquad \alpha_2 = L_3 - L_4, \qquad \alpha_3 = L_4,$$
$$\alpha_4 = \frac{L_1 - L_2 - L_3 - L_4}{2};$$

$$R^+ = \{L_i\} \cup \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j} \cup \left\{\frac{L_1 \pm L_2 \pm L_3 \pm L_4}{2}\right\}.$$

In particular, (F_4) has 24 positive roots.

(E_6)
$$\alpha_1 = \frac{L_1 - L_2 - L_3 - L_4 - L_5 + \sqrt{3}L_6}{2}, \qquad \alpha_2 = L_1 + L_2,$$

$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2,$$
$$\alpha_5 = L_4 - L_3, \qquad \alpha_6 = L_5 - L_4;$$
$$R^+ = \{L_i + L_j\}_{i<j\le 5} \cup \{L_i - L_j\}_{j<i\le 5}$$
$$\cup \left\{\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4 \pm L_5 + \sqrt{3}L_6}{2}\right\}_{\text{number of minus signs even}}.$$

(E_6) has 36 positive roots.

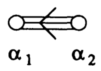
(E_7) $$\alpha_1 = \frac{L_1 - L_2 - \cdots - L_6 + \sqrt{2}L_7}{2}, \qquad \alpha_2 = L_1 + L_2,$$
$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2, \qquad \alpha_5 = L_4 - L_3,$$
$$\alpha_6 = L_5 - L_4, \qquad \alpha_7 = L_6 - L_5;$$
$$R^+ = \{L_i + L_j\}_{i<j\le 6} \cup \{L_i - L_j\}_{j<i\le 6} \cup \{\sqrt{2}L_7\}$$
$$\cup \left\{\frac{\pm L_1 \pm L_2 \pm \cdots \pm L_6 + \sqrt{2}L_7}{2}\right\}_{\text{number of minus signs odd}}.$$

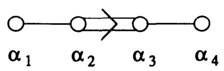
Thus, (E_7) has 63 positive roots.

(E_8) $$\alpha_1 = \frac{L_1 - L_2 - \cdots - L_7 + L_8}{2}, \qquad \alpha_2 = L_1 + L_2,$$
$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2, \qquad \alpha_5 = L_4 - L_3,$$
$$\alpha_6 = L_5 - L_4, \qquad \alpha_7 = L_6 - L_5, \qquad \alpha_8 = L_7 - L_6.$$
$$R^+ = \{L_i + L_j\}_{i<j\le 8} \cup \{L_i - L_j\}_{j<i\le 8}$$
$$\cup \left\{\frac{\pm L_1 \pm L_2 \pm \cdots \pm L_7 + L_8}{2}\right\}_{\text{number of minus signs even}}.$$

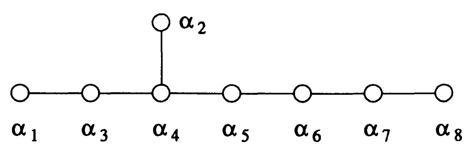
(E_8) has 120 positive roots.

For (G_2) and (F_4) the simple roots are listed in order reading from left to right in their Dynkin diagrams





as in the classical series (A_n)–(D_n). For (E_8), the numbering is



while those for (E_7) and (E_6) are obtained by removing the last one or two nodes. Note that, given the root system of (E_8), we can find the root system of (E_7) or (E_6) by taking the subspace spanned by the first seven or six simple roots.

Exercise 21.16*. (a) Verify the above lists of roots.
(b) In each case, calculate the corresponding fundamental weights.

Exercise 21.17*. Show that no two of the root systems of (A_n)–(E_8) are isomorphic, and deduce that the Dynkin diagram of a root system is independent of choice of positive roots.

A more satisfying reason for the last fact is the observation that any two choices of positive roots differ by an element of the Weyl group—the group generated by reflections W_α in the simple roots. This can be seen directly for each of the diagrams (A_n)–(E_8); for a general proof that two choices differ by an element of the Weyl group, see Proposition D.29.

We should mention here another way of conveying the data of a Dynkin diagram. This is simply the $n \times n$ matrix of integers $(n_{i,j} = n_{\alpha_i \alpha_j})$, where we take $n_{i,i} = 2$; it is called the *Cartan matrix* of the Dynkin diagram (or of the Lie algebra). Thus, for example, the Cartan matrix of (A_n) is

$$\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 \end{pmatrix}.$$

These matrices pop up remarkably often, in a variety of seemingly unrelated areas of mathematics. They will not play a major role in the present text, but the reader has probably encountered them already in one form or another, and will probably do so again.

Exercise 21.18*. Compute the Cartan matrix, and its determinant, for each Dynkin diagram.

The next task is to see how the root system determines the Lie algebra. We concentrate on the uniqueness, since there are other ways to see the existence; indeed, for all but the five exceptions we have already seen the Lie algebras. We will describe several approaches to this problem, starting with a straightforward and computational method and finishing with a slick but abstract approach.

Assume as before that $\mathfrak{g}$ is a simple Lie algebra, with a chosen Cartan subalgebra $\mathfrak{h}$ and decomposition of the roots R into positive and negative roots; let $\alpha_1, \dots, \alpha_n$ be the simple roots. The Dynkin diagram information is the knowledge of (α_i, α_j) for all $i \neq j$. Let $H_i = H_{\alpha_i}$ be the corresponding basis of $\mathfrak{h}$, defined by the rule we have seen in Lecture 14: if $\{T_i\}$ is the basis corresponding via the Killing form to $\{\alpha_i\}$, set $H_i = 2T_i/(\alpha_i, \alpha_i)$.

Choose any nonzero element X_i in the root space $\mathfrak{g}_{\alpha_i}$, for $1 \leq i \leq n$. This determines elements Y_i in $\mathfrak{g}_{-\alpha_i}$ such that $[X_i, Y_i] = H_i$. We claim first that these $3n$ elements $\{H_i, X_i, Y_i\}$ generate $\mathfrak{g}$ as a Lie algebra. This follows from

Claim 21.19. *If α, β, and $\alpha + \beta$ are roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

PROOF. Again look at the α-string through $\mathfrak{g}_\beta$, i.e., $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$. This is an irreducible representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$, since all the terms are one dimensional (this follows from the fact that no $\beta + k\alpha$ can be zero, given that $\beta \neq \pm\alpha$). But now if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, $\bigoplus_{k \leq 0} \mathfrak{g}_{\beta+k\alpha}$ would be a nontrivial subrepresentation. □

For each positive root β, we have seen that can write β as a sum of simple roots $\beta = \alpha_{i_1} + \cdots + \alpha_{i_r}$ such that each of the sums $\alpha_{i_1} + \cdots + \alpha_{i_s}$ is a root, $1 \leq s \leq r$. If we choose such a presentation for each β, and set

$$X_\beta = [X_{i_r}, [X_{i_{r-1}}, \dots, [X_{i_2}, X_{i_1}] \dots]]$$

and

$$Y_\beta = [Y_{i_r}, [Y_{i_{r-1}}, \dots, [Y_{i_2}, Y_{i_1}] \dots]],$$

then the collection

$$\{H_i, 1 \leq i \leq n; X_\beta, Y_\beta, \beta \in R^+\} \tag{21.20}$$

forms a basis for $\mathfrak{g}$. Note that if β is not simple, there is no reason to expect $[X_\beta, Y_\beta]$ to be the distinguished element H_β in $\mathfrak{h}$.

We want to show that the multiplication table for these basis elements is completely determined by the Dynkin diagram. The main difficulty is that the ordering of the simple roots in the above expression for β may not be unique. For example, suppose

$$\beta = (\alpha_1 + \alpha_2) + \alpha_3 = (\alpha_2 + \alpha_3) + \alpha_1,$$

with $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ roots. We must compare $[X_3, [X_2, X_1]]$ with $[X_1, [X_3, X_2]]$. In fact, they must be negatives of each other. For, by Jacobi, we have

$$[X_1, [X_3, X_2]] = -[X_3, [X_2, X_1]] - [X_2, [X_1, X_3]] = -[X_3, [X_2, X_1]],$$

noting that $[X_1, X_3] = 0$ since $\alpha_1 + \alpha_3$ cannot be a root, e.g., by step (ii) of the preceding section.

For any sequence $I = (i_1, \dots, i_r)$, $1 \leq i_j \leq n$, set

$$\alpha_I = \alpha_{i_1} + \cdots + \alpha_{i_r},$$
$$X_I = [X_{i_r}, [X_{i_{r-1}}, \ldots, [X_{i_2}, X_{i_1}] \ldots]],$$
$$Y_I = [Y_{i_r}, [Y_{i_{r-1}}, \ldots, [Y_{i_2}, Y_{i_1}] \ldots]].$$

Call I *admissible* if each partial sum $\alpha_{i_1} + \cdots + \alpha_{i_s}$ is a root, $1 \le s \le r$; note that I is admissible exactly when X_I is not zero.

Lemma 21.21. *If I and J are two admissible sequences for which $\alpha_I = \alpha_J$, then there is a nonzero rational number q determined by I, J, and the Dynkin diagram, such that $X_J = q \cdot X_I$.*

Proof. Let $k = i_r$ be the last entry in I. If $j_r = k$ as well, the result follows by induction on r. We reduce the general case to this case, by maneuvering to replace j_r by k. We have first

$$X_J = q_1 \cdot [X_k, [Y_k, X_J]],$$

with q_1 a nonzero rational number depending only on J, k, and the Dynkin diagram, since $\alpha_J - \alpha_k = \alpha_I - \alpha_k$ is a root; the point is that we know how $\mathfrak{s}_{\alpha_k} \cong \mathfrak{sl}_2$ acts on the α_k-string through α_J as soon as we know the length of the string, and this is Dynkin diagram information. Next, let s be the largest integer such that $j_s = k$. Then

$$[Y_k, X_J] = [X_{j_r}, \ldots [X_{j_{s+1}}, [Y_k, [X_k, X_K]]] \ldots],$$

where $K = (j_1, \ldots, j_{s-1})$, since $[Y_k, [X_i, Z]] = [X_i, [Y_k, Z]]$ when $i \neq k$. Finally,

$$[Y_k, [X_k, X_K]] = q_2 \cdot X_K,$$

with q_2 a nonzero rational number depending only on K, k, and the Dynkin diagram, since $\alpha_K + \alpha_k$ is a root. Combining these three equations, we get

$$X_J = q_1 q_2 \cdot [X_k, [X_{j_r}, \ldots [X_{j_{s+1}}, X_K] \ldots]],$$

which suffices since the sequence for the term on the right ends in the same integer k as I. □

Proposition 21.22. *The bracket of any two basis elements in* (21.20) *is a rational multiple of another basis element, that multiple determined from the Dynkin diagram.*

Proof. This is clear for brackets of an H_i with any basis element. Lemma 21.21 handles brackets of the form $[X_I, X_J]$, and those involving only Y's are similar. For brackets $[Y_I, X_J]$, it suffices inductively to compute $[Y_k, X_J]$ as a rational multiple of some X_K, with K shorter than J (or of H_k if J has one term); but this was worked out in the proof of the lemma. □

Exercise 21.23*. (i) Show that in (G_2) each positive root can be written in only one way as a sum of simple roots, up to the order of the first two roots.

(ii) Work out the multiplication table from the Dynkin diagram. (iii) Verify that the result is indeed a Lie algebra, which is (visibly) simple.

This exercise will be worked out in detail to start the next lecture. Of course, there is nothing but lack of time to keep us from verifying that the other four exceptional Dynkin diagrams do lead, by the same prescription, to honest Lie algebras, but doing it by hand gets pretty laborious, and we will describe some of the other methods available.

The fact that the multiplication table can be defined with rational coefficients becomes important when one wants to reduce them modulo prime numbers, which we will not discuss here. The fact that they can be taken to be real, on the other hand, will come up later, when we discuss real forms of complex Lie algebras and groups.

There is a more general and elegant way to proceed, given by Serre [Se3]. Write n_{ij} in place of $n_{\alpha_i\alpha_j}$. Form the free Lie algebra on generators

$$H_1, \ldots, H_n, X_1, \ldots, X_n, Y_1, \ldots, Y_n,$$

i.e., form the free (tensor) algebra with this basis, and divide modulo by the relations $[A, B] + [B, A] = 0$ and the Jacobi relation. Then take this free Lie algebra, and divide by the relations

$$[H_i, H_j] = 0 \text{ (all } i, j); \qquad [X_i, Y_i] = H_i \text{ (all } i); \qquad [X_i, Y_j] = 0 \ (i \neq j);$$
$$[H_i, X_j] = n_{ji} X_j \text{ (all } i, j); \qquad [H_i, Y_j] = -n_{ji} Y_j \text{ (all } i, j);$$

and, for all $i \neq j$,

$$\begin{aligned}
[X_i, X_j] &= 0, & [Y_i, Y_j] &= 0 & &\text{if } n_{ij} = 0;\\
[X_i, [X_i, X_j]] &= 0, & [Y_i, [Y_i, Y_j]] &= 0 & &\text{if } n_{ij} = -1;\\
[X_i, [X_i, [X_i, X_j]]] &= 0, & [Y_i, [Y_i, [Y_i, Y_j]]] &= 0 & &\text{if } n_{ij} = -2;\\
[X_i, [X_i, [X_i, [X_i, X_j]]]] &= 0, & [Y_i, [Y_i, [Y_i, [Y_i, Y_j]]]] &= 0 & &\text{if } n_{ij} = -3.
\end{aligned}$$

Exercise 21.24. Verify that if one starts with a semisimple Lie algebra with a given Dynkin diagram, the above equations must hold.

Serre shows ([Se3, Chap. VI App.], cf. [Hu1 §18]) that the resulting Lie algebra is a finite-dimensional semisimple Lie algebra, with Cartan subalgebra generated by $H_1, \ldots, H_n$ and given root system. In particular, this includes a proof of the existence of all the simple Lie algebras.

Here is a third approach to uniqueness. Suppose $\mathfrak{g}$ and $\mathfrak{g}'$, with given Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$, and choice of positive roots, have isomorphic root systems. There is an isomorphism $\mathfrak{h} \to \mathfrak{h}'$, taking corresponding H_i to H_i'. Choose arbitrarily nonzero vectors X_i and X_i' in the root spaces of $\mathfrak{g}$ and $\mathfrak{g}'$ corresponding to the simple roots.

Claim 21.25. *There is a unique isomorphism from* $\mathfrak{g}$ *to* $\mathfrak{g}'$ *extending the isomorphism of* $\mathfrak{h}$ *with* $\mathfrak{h}'$, *and mapping* X_i *to* X_i' *for all* i.

Proof. The uniqueness of the isomorphism is easy: the resulting map is determined on the Y_i by $\mathfrak{sl}_2$ considerations, and the H_i, X_i, and Y_i generate $\mathfrak{g}$. For the existence of the isomorphism consider the subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g} \oplus \mathfrak{g}'$ generated by $\tilde{H}_i = H_i \oplus H_i'$, $\tilde{X}_i = X_i \oplus X_i'$, and $\tilde{Y}_i = Y_i \oplus Y_i'$. It suffices to prove that the two projections from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphisms. The kernel of the second projection is $\mathfrak{k} \oplus 0$, where $\mathfrak{k}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{k}$ is either 0, as required, or $\mathfrak{k} = \mathfrak{g}$. In the latter case, we must have $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$.

To see that this is impossible, consider a maximal positive root β, take nonzero vectors X_β, X_β' in the corresponding root spaces, and set $\tilde{X}_\beta = X_\beta \oplus X_\beta'$, a highest weight vector in $\tilde{\mathfrak{g}}$. Let W be the subspace of $\tilde{\mathfrak{g}}$ obtained by successively applying all $\tilde{Y}_i$'s. Then W is a proper subspace of $\tilde{\mathfrak{g}}$, since its weight space W_β corresponding to β is one dimensional. By the argument we have seen several times, $\tilde{\mathfrak{g}}$ preserves W. Now if $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$, W would be an ideal in $\mathfrak{g} \oplus \mathfrak{g}'$, and this would force $X_\beta \oplus 0$ to belong to W, making W_β two dimensional again. □

To finish this story, we should show that the simple Lie algebras corresponding to two different Dynkin diagrams cannot be isomorphic, i.e., that the two choices made in going from a semisimple Lie algebra to Dynkin diagram do not change the answer. The general facts are:

(1) Any two Cartan subalgebras of a semisimple Lie algebra are conjugate, i.e., there is an inner automorphism by an element in the corresponding adjoint group, which takes one into the other.
(2) Any two decompositions of a root system into positive and negative roots differ by an element of the Weyl group.

These are standard facts which are proved in Appendix D. Both statements are subsumed in the fact that any two *Borel subalgebras* of a semisimple Lie algebra are conjugate, a Borel subalgebra being the subspace spanned by the Cartan subalgebra and the root spaces $\mathfrak{g}_\alpha$ for positive α. For those readers who crave logical completeness but do not want to go through so much general theory, we observe that most possible coincidences can be ruled out by such simple considerations as computing dimensions, and others can be ruled out by simple ad hoc methods, cf. Exercise 21.17.

Finally, we must also prove the "existence theorem": that there is a simple Lie algebra for each Dynkin diagram. Serre's theorem quoted above gives a unified proof of existence. But we have seen and studied the Lie algebras for the classical cases (A_n)–(D_n), and it is more in keeping with the spirit of these lectures to at least try to see the five exceptions explicitly. This is the subject of the next lecture.

LECTURE 22

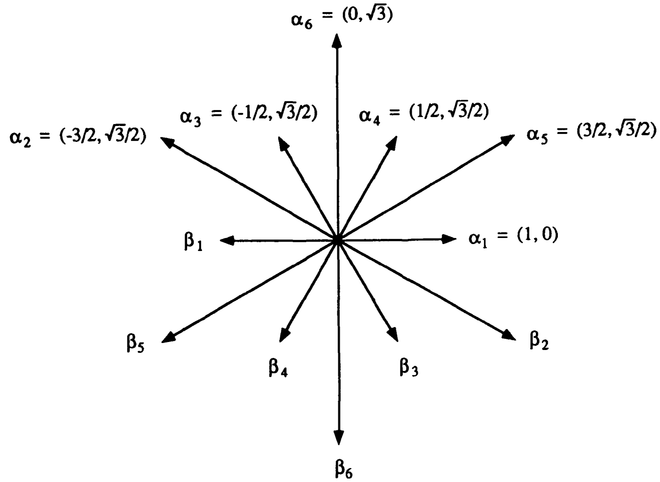
$\mathfrak{g}_2$ and Other Exceptional Lie Algebras

This lecture is mainly about $\mathfrak{g}_2$, with just enough discussion of the algebraic constructions of the other exceptional Lie algebras to give the reader a sense of their complexity. $\mathfrak{g}_2$, being only 14-dimensional, is different: we can reasonably carry out in practice the process described in §21.3 to arrive at an explicit description of the algebra by specifying a basis and all pairwise products; we do this in §22.1 and verify in §22.2 that the result really is a Lie algebra. In §22.3 we analyze the representations of $\mathfrak{g}_2$, and arrive in particular at another description of $\mathfrak{g}_2$: it is the algebra of endomorphisms of a seven-dimensional vector space preserving a general trilinear form. (Note that §22.3 may be read independently of either §22.1, §21.2, or §21.3.) Finally, in the fourth section we will sketch some of the more abstract (i.e., coordinate free) approaches to the construction of the five exceptional Lie algebras. While the first two sections are completely elementary, the constructions given in §22.4 involve some fairly serious algebra.

§22.1. Construction of $\mathfrak{g}_2$ from Its Dynkin Diagram

In this section we will carry out explicitly the process described in the preceding section for the Dynkin diagram (G_2), constructing in this way a Lie algebra $\mathfrak{g}_2$ with diagram (G_2) (and in particular proving its existence).

The first step is to find the root system from the Dynkin diagram. In the case of $\mathfrak{g}_2$ this is immediate; we may draw the root system $R \subset \mathfrak{h}^*$ associated to the diagram G_2 as follows:



Here the positive roots are denoted α_i, with α_1 and α_2 the simple roots. The coordinate system here has no particular significance (in particular, recall that the configuration of roots α_i and β_i is determined only up to a real scalar), but is convenient for calculating inner products. Note that the Weyl group is the dihedral group generated by rotation through an angle of $\pi/3$ and reflection in the horizontal; the Weyl chamber associated to the choice of ordering of the roots given is the cone between the roots α_6 and α_4.

As indicated in the preceding section, we start by letting X_1 be any eigenvector for the action of $\mathfrak{h}$ with eigenvalue α_1, and X_2 any eigenvector for the action of $\mathfrak{h}$ with eigenvalue α_2. We similarly let Y_1 and Y_2 be eigenvectors with eigenvalues β_1 and β_2 and set

$$H_1 = [X_1, Y_1] \quad \text{and} \quad H_2 = [X_2, Y_2].$$

We can choose Y_1 and Y_2 so that the elements $H_i \in \mathfrak{h}$ satisfy $\alpha_1(H_1) = \alpha_2(H_2) = 2$, i.e.,

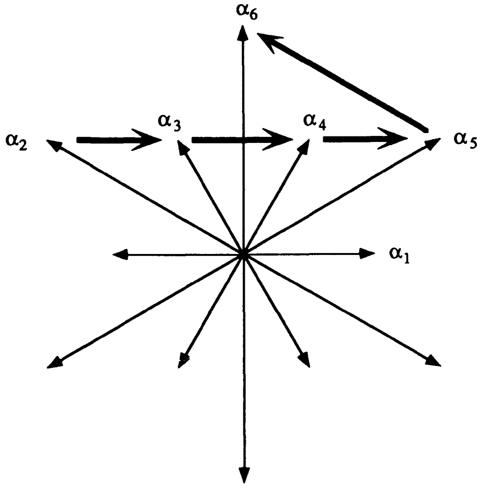
$$[H_1, X_1] = 2 \cdot X_1 \quad \text{and} \quad [H_2, X_2] = 2 \cdot X_2.$$

It follows that

$$[H_1, Y_1] = -2 \cdot Y_1 \quad \text{and} \quad [H_2, Y_2] = -2 \cdot Y_2,$$

i.e., H_i, X_i, and Y_i span a subalgebra $\mathfrak{s}_{\alpha_i} \cong \mathfrak{sl}_2\mathbb{C}$, with H_i, X_i, and Y_i a normalized basis for this copy of $\mathfrak{sl}_2\mathbb{C}$.

Now, it is clear from the diagram above that there is a unique way of writing each positive root α_i as a sum of simple roots $\alpha_{i_1} + \cdots + \alpha_{i_k}$ so that the partial sums $\alpha_{i_1} + \cdots + \alpha_{i_l}$ are roots for each $l \le k$ (modulo exchanging the first two terms): we go through the root system by the path



i.e., we write

$$\alpha_3 = \alpha_1 + \alpha_2,$$

$$\alpha_4 = \alpha_1 + \alpha_3 = \alpha_1 + \alpha_1 + \alpha_2,$$

$$\alpha_5 = \alpha_1 + \alpha_4 = \alpha_1 + \alpha_1 + \alpha_1 + \alpha_2,$$

$$\alpha_6 = \alpha_2 + \alpha_5 = \alpha_2 + \alpha_1 + \alpha_1 + \alpha_1 + \alpha_2.$$

According to the general recipe, this means we now set

$$X_3 = [X_1, X_2], \qquad X_4 = [X_1, X_3],$$
$$X_5 = [X_1, X_4], \qquad X_6 = [X_2, X_5],$$

and define $Y_3, \ldots, Y_6$ similarly. The elements $H_1, H_2, X_1, \ldots, X_6, Y_1, \ldots, Y_6$ then form a basis for the 14-dimensional $\mathfrak{g}_2$, with H_1 and H_2 a basis for $\mathfrak{h}$, X_i a generator of the eigenspace $\mathfrak{g}_{\alpha_i}$, and Y_i a generator of $\mathfrak{g}_{\beta_i}$ for $i = 1, \ldots, 6$.

The task at hand now is to write down the multiplication table for $\mathfrak{g}_2$ in terms of this basis. Of course, some products are already known: we know, for example, that H_i, X_i, and Y_i form a normalized basis for $\mathfrak{sl}_2\mathbb{C}$ for $i = 1, 2$, and we have the relations defining $X_3, \ldots, X_6$ and $Y_1, \ldots, Y_6$ above. In addition, since we know that the product $[X_i, X_j]$ lies in the root space $\mathfrak{g}_{\alpha_i+\alpha_j}$ for each i and j, we see immediately that $[X_i, X_j] = 0$ whenever $\alpha_i + \alpha_j$ is not a root. We deduce that

$$[X_1, X_5] = [X_1, X_6] = [X_2, X_3] = [X_2, X_4] = [X_2, X_6] = [X_3, X_5]$$
$$= [X_3, X_6] = [X_4, X_5] = [X_4, X_6] = [X_5, X_6] = 0,$$

and likewise

$$[Y_1, Y_5] = [Y_1, Y_6] = [Y_2, Y_3] = [Y_2, Y_4] = [Y_2, Y_6] = [Y_3, Y_5]$$
$$= [Y_3, Y_6] = [Y_4, Y_5] = [Y_4, Y_6] = [Y_5, Y_6] = 0.$$

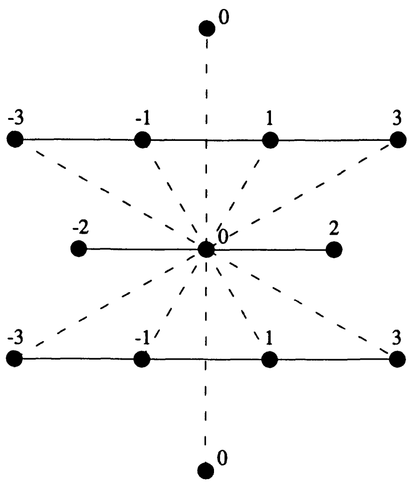
Similarly, we know that $[X_i, Y_j] = 0$ whenever $\alpha_i + \beta_j = \alpha_i - \alpha_j$ is not a root; this tells us as well that

$$\begin{aligned}[][X_1, Y_2] &= [X_1, Y_6] = [X_2, Y_1] = [X_2, Y_4] = [X_2, Y_5] = [X_3, Y_5]\\ &= [X_4, Y_2] = [X_5, Y_2] = [X_5, Y_3] = [X_6, Y_1] = 0.\end{aligned}$$

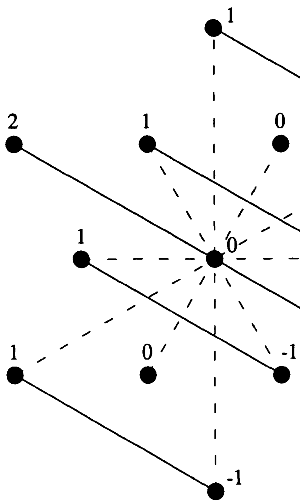
The multiplication table thus far looks like

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	*	*	*	*	*	*	*	*	*	*
H_2		*	*	$2X_2$	$-2Y_2$	*	*	*	*	*	*	*	*
X_1			H_1	X_3	0	X_4	*	X_5	*	0	*	0	0
Y_1				0	Y_3	*	Y_4	*	Y_5	*	0	0	0
X_2					H_2	0	*	0	0	X_6	0	0	*
Y_2						*	0	0	0	0	Y_6	*	0
X_3							*	*	*	0	0	0	*
Y_3								*	*	0	0	*	0
X_4									*	0	*	0	*
Y_4										*	0	*	0
X_5											*	0	*
Y_5												*	0
X_6													*

The next thing to do is to describe the action of H_1 and H_2 on the various vectors X_i and Y_i. This can be done using the inner product on $\mathfrak{h}$, but it is perhaps simpler to go back to the basic idea of restriction to the subalgebras $\mathfrak{s}_{\alpha_1}$ and $\mathfrak{s}_{\alpha_2}$. For example, if we want to determine the action of H_1 on the various X_i, consider how the algebra $\mathfrak{g} = \mathfrak{h} \bigoplus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{\beta_i})$ decomposes as a representation of $\mathfrak{s}_{\alpha_1}$:



We get two trivial representations (the spans of X_6 and Y_6, as already noted); one copy of the adjoint representation $\mathrm{Sym}^2 V$ (the subalgebra $\mathfrak{s}_{\alpha_1}$ itself) spanned by X_1, Y_1, and H_1; and two copies of the irreducible four-dimensional representation $\mathrm{Sym}^3 V$ spanned by X_2, X_3, X_4, and X_5 and Y_5, Y_4, Y_3, and Y_2. In particular, it follows that X_2, X_3, X_4, and X_5 are eigenvectors for the action of H_1 with eigenvalues of -3, -1, 1, and 3, respectively; and likewise Y_5, Y_4, Y_3, and Y_2 are eigenvectors with eigenvalues -3, -1, 1, and 3. In similar fashion, we consider the decomposition of $\mathfrak{g}$ under the action of $\mathfrak{s}_{\alpha_2} = \mathbb{C}\{H_2, X_2, Y_2\}$: diagrammatically, this looks like



Here we have two trivial representations, spanned by X_4 and Y_4, one adjoint ($\mathfrak{s}_{\alpha_2}$ itself), and four copies of the standard two-dimensional representation V, spanned by X_6 and X_5, X_3 and X_1, Y_1 and Y_3, and Y_5 and Y_6. It follows that X_6, X_3, Y_1, and Y_5 are eigenvectors for the action of H_2 with eigenvalue 1, and likewise X_5, X_1, Y_3, and Y_6 are eigenvectors with eigenvalue -1.

Including this information, we can fill in the top two rows of the multiplication table:

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$

Decomposing $\mathfrak{g}_2$ according to the action of $\mathfrak{s}_{\alpha_1}$ and $\mathfrak{s}_{\alpha_2}$ gives us information about the action of X_1, X_2, Y_1, and Y_2 on the other basis vectors as well. For example, we saw a moment ago that X_5 and X_6 together span a subrepresentation of $\mathfrak{g}_2$ under the action of $\mathfrak{s}_{\alpha_2}$, with $\mathrm{ad}(X_2)$ carrying X_5 to X_6. It follows from this that $\mathrm{ad}(Y_2)$ must carry X_6 back to X_5: we have

$$\begin{aligned}\mathrm{ad}(Y_2)(X_6) &= \mathrm{ad}(Y_2)\,\mathrm{ad}(X_2)(X_5)\\ &= \mathrm{ad}(X_2)\,\mathrm{ad}(Y_2)(X_5) - \mathrm{ad}([X_2, Y_2])(X_5)\\ &= 0 - \mathrm{ad}(H_2)(X_5) = X_5.\end{aligned}$$

Similarly, since $\mathrm{ad}(X_2)$ carries X_1 into $-X_3$, which together with X_1 spans a copy of the standard two-dimensional representation of $\mathfrak{s}_{\alpha_2} \cong \mathfrak{sl}_2\mathbb{C}$, it follows that $\mathrm{ad}(Y_2)$ will carry $-X_3$ back to X_1. Likewise from the fact that $\mathrm{ad}(Y_2)$ carries Y_1 to $-Y_3$ we see that $\mathrm{ad}(Y_2)(Y_3) = -Y_1$, and since $\mathrm{ad}(Y_2)$: $Y_5 \mapsto Y_6$, $\mathrm{ad}(X_2)$: $Y_6 \mapsto Y_5$.

We can in the same way use the action of $\mathfrak{s}_{\alpha_1}$ to determine the values of $\mathrm{ad}(X_1)$ and $\mathrm{ad}(Y_2)$ on various basis vectors, though because the representation of $\mathfrak{s}_{\alpha_1}$ on $\mathfrak{g}_2$ has larger-dimensional components this is slightly more complicated. To begin with, consider the representation of $\mathfrak{s}_{\alpha_1}$ on the subspace spanned by X_2, X_3, X_4, and X_5. We know that $\mathrm{ad}(X_1)$ carries X_2 to X_3, and since X_2 is an eigenvector for the action of the commutator $[X_1, Y_1] = H_1$ with eigenvalue -3, it follows that $\mathrm{ad}(Y_1)$ must carry X_3 to $3X_2$: we have

$$\begin{aligned}\mathrm{ad}(Y_1)(X_3) &= \mathrm{ad}(Y_1)\,\mathrm{ad}(X_1)(X_2)\\ &= \mathrm{ad}(X_1)\,\mathrm{ad}(Y_1)(X_2) - \mathrm{ad}([X_1, Y_1])(X_2)\\ &= 0 - \mathrm{ad}(H_1)(X_2) = 3X_2.\end{aligned}$$

Using this, we can next determine the action of Y_1 on X_4:

$$\begin{aligned}\mathrm{ad}(Y_1)(X_4) &= \mathrm{ad}(Y_1)\,\mathrm{ad}(X_1)(X_3)\\ &= \mathrm{ad}(X_1)\,\mathrm{ad}(Y_1)(X_3) - \mathrm{ad}(H_1)(X_3)\\ &= \mathrm{ad}(X_1)(3X_2) + X_3 = 4X_3,\end{aligned}$$

and we calculate likewise that $\mathrm{ad}(Y_1)(X_5) = 3X_4$. Analogously, knowing that $\mathrm{ad}(Y_1)$ carries Y_2 to Y_3 to Y_4 to Y_5 yields the information that $\mathrm{ad}(X_1)$ must carry Y_3, Y_4, and Y_5 to $3Y_2$, $4Y_3$ and, $3Y_4$, respectively. Including all this information in the chart, the next four rows of our multiplication table are

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
X_1			H_1	X_3	0	X_4	$3Y_2$	X_5	$4Y_3$	0	$3Y_4$	0	0
Y_1				0	Y_3	$3X_2$	Y_4	$4X_3$	Y_5	$3X_4$	0	0	0
X_2					H_2	0	$-Y_1$	0	0	X_6	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	X_5	0

We next have to find the commutators of the basis elements X_i and Y_j for $i, j \geq 3$. We cannot do this by looking at the action of the subalgebras generated by X_i and Y_i, since for $i \geq 3$ we do not know the commutator $[X_i, Y_i]$. Rather, the way to do this is outlined in the general proof in the preceding section: we just use the expression of the X_i and Y_j as brackets of the generators X_1, X_2, Y_1, and Y_2 to reduce the problem to brackets with these generators, which we now know. Thus, for example, the first unknown entry in the table at present is the bracket $[X_3, Y_3]$. We calculate this by writing X_3 as $[X_1, X_2]$, so that

$$\begin{aligned}
\operatorname{ad}(X_3)(Y_3) &= \operatorname{ad}([X_1, X_2])(Y_3)\\
&= \operatorname{ad}(X_1)\operatorname{ad}(X_2)(Y_3) - \operatorname{ad}(X_2)\operatorname{ad}(X_1)(Y_3)\\
&= \operatorname{ad}(X_1)(-Y_1) - \operatorname{ad}(X_2)(3Y_2)\\
&= -H_1 - 3H_2.
\end{aligned}$$

Likewise, to evaluate $[X_3, X_4]$ we have

$$\begin{aligned}
\operatorname{ad}(X_3)(X_4) &= \operatorname{ad}([X_1, X_2])(X_4)\\
&= \operatorname{ad}(X_1)\operatorname{ad}(X_2)(X_4) - \operatorname{ad}(X_2)\operatorname{ad}(X_1)(X_4)\\
&= -\operatorname{ad}(X_2)(X_5) = -X_6.
\end{aligned}$$

In this way, we can evaluate all brackets with X_3; knowing these, we can reduce any bracket with X_4 to one involving X_1 and X_3 by writing $X_4 = [X_1, X_3]$, and so on. Continuing in this way, we may complete our multiplication table:

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	X_4	$3Y_2$	X_5	$4Y_3$	0	$3Y_4$	0	0
Y_1				0	Y_3	$3X_2$	Y_4	$4X_3$	Y_5	$3X_4$	0	0	0
X_2					H_2	0	$-Y_1$	0	0	X_6	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	X_5	0
X_3							$-H_1$ $-3H_2$	$-X_6$	$4Y_1$	0	0	0	$3Y_4$
Y_3								$4X_1$	$-Y_6$	0	0	$3X_4$	0
X_4									$8H_1$ $+12H_2$	0	$-12Y_1$	0	$12Y_3$
Y_4										$-12X_1$	0	$12X_3$	0
X_5											$-36H_1$ $-36H_2$	0	$36Y_2$
Y_5												$36X_2$	0
X_6													$36H_1$ $+72H_2$

Of course, in retrospect we see that the basis we have chosen is far from the most symmetric one possible: for example, if we divided X_4 and Y_4 by 2 and X_5, X_6, Y_5, and Y_6 by 6, and changed the signs of X_5 and Y_3, the form of the table would be

Table 22.1

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	$2X_4$	$-3Y_2$	$-3X_5$	$-2Y_3$	0	Y_4	0	0
Y_1				0	$-Y_3$	$3X_2$	$-2Y_4$	$2X_3$	$3Y_5$	$-X_4$	0	0	0
X_2					H_2	0	Y_1	0	0	$-X_6$	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	$-X_5$	0
X_3							H_1+3H_2	$-3X_6$	$2Y_1$	0	0	0	Y_4
Y_3								$-2X_1$	$3Y_6$	0	0	$-X_4$	0
X_4									$2H_1+3H_2$	0	$-Y_1$	0	$-Y_3$
Y_4										X_1	0	X_3	0
X_5											H_1+H_2	0	$-Y_2$
Y_5												X_2	0
X_6													H_1+2H_2

There was another good reason for these changes: now each of the brackets $[X_i, Y_i]$ will be the distinguished element of $\mathfrak{h}$ corresponding to the root α_i. If we denote this element by H_i, then we read off from the table that

$$\begin{aligned} H_3 &= H_1 + 3H_2, \qquad & H_4 &= 2H_1 + 3H_2, \\ H_5 &= H_1 + H_2, \qquad & H_6 &= H_1 + 2H_2, \end{aligned} \tag{22.2}$$

and

$$H_i = [X_i, Y_i], \qquad [H_i, X_i] = 2X_i, \qquad [H_i, Y_i] = -2Y_i, \tag{22.3}$$

for $i = 1, 2, 3, 4, 5, 6$.

§22.2. Verifying That $\mathfrak{g}_2$ Is a Lie Algebra

The calculation of the preceding section gives a complete description of what the Lie algebra $\mathfrak{g}_2$ must look like, but there is still some work to be done: unless we know that there is a Lie algebra with diagram (G_2), we do not know that the above multiplication table defines a Lie algebra, let alone a simple one. In fact, the simplicity is not much of a problem (cf. Exercise 14.34), but to know that it is a Lie algebra requires knowing that the Jacobi identity is valid. One could simply check this from the table for all $\binom{14}{3}$ triples of elements from the basis, a rather uninviting task.

There is another way, which gives more structure to the preceding calculations, and which will give a clue for possible constructions of other Lie algebras. The root diagram for (G_2) is made up of two hexagons, one with long arrows, the other with short. This suggests that we should find a copy of the corresponding Lie algebra $\mathfrak{sl}_3\mathbb{C}$ inside $\mathfrak{g}_2$. The subspace spanned by $\mathfrak{h}$ and the root spaces corresponding to the six longer roots is clearly closed under brackets, so is the obvious candidate. The long roots are α_5, α_2, and $\alpha_6 = \alpha_5 + \alpha_2$, and their inverses. So we define $\mathfrak{g}_0$ to be the subspace spanned by the corresponding vectors:

$$\mathfrak{g}_0 = \mathbb{C}\{H_5, H_2, X_5, Y_5, X_2, Y_2, X_6, Y_6\}.$$

The multiplication table for $\mathfrak{g}_0$ is read off from Table 22.1:

	H_2	X_5	Y_5	X_2	Y_2	X_6	Y_6
H_5	0	$2X_5$	$-2Y_5$	$-X_2$	Y_2	X_6	$-Y_6$
H_2		$-X_5$	Y_5	$2X_2$	$-2Y_2$	X_6	$-Y_6$
X_5			H_5	X_6	0	0	$-Y_2$
Y_5				0	$-Y_6$	X_2	0
X_2					H_2	0	Y_5
Y_2						$-X_5$	0
X_6							$H_5 + H_2$

This is exactly the multiplication table for $\mathfrak{sl}_3\mathbb{C}$, with its standard basis (in the same order):

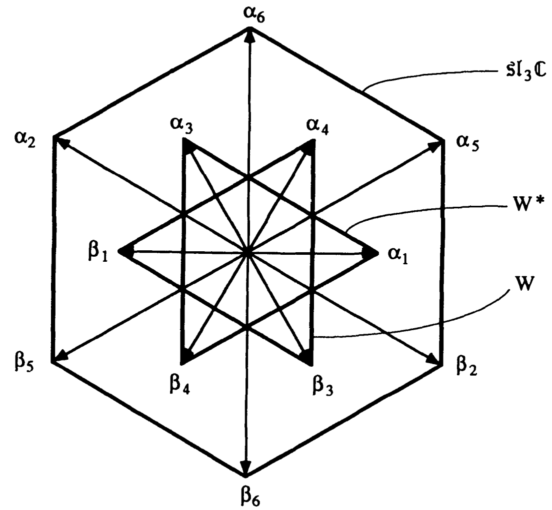
$$\mathfrak{sl}_3\mathbb{C} = \mathbb{C}\{E_{1,1} - E_{2,2}, E_{2,2} - E_{3,3}, E_{1,2}, E_{2,1}, E_{2,3}, E_{3,2}, E_{1,3}, E_{3,1}\}.$$

So we have determined an isomorphism

$$\mathfrak{g}_0 \cong \mathfrak{sl}_3\mathbb{C}.$$

(Note right away that this verifies the Jacobi identity for triples taken from $\mathfrak{g}_0$.)

The rest of the Lie algebra must be a representation of the subalgebra $\mathfrak{g}_0 \cong \mathfrak{sl}_3\mathbb{C}$, and we know what this must be: the smaller hexagon is the union of the two triangles which are the weight diagrams for the standard representation of $\mathfrak{sl}_3$ and its dual, which we denote here by W and W^*; W is the sum of the root spaces for α_4, β_1, and β_3, while W^* is the sum of those for β_4, α_1, and α_3.



Again, a look at the table shows that the vectors X_4, Y_1, and Y_3 form a basis for $W = \mathbb{C}^3$ that corresponds to the standard basis e_1, e_2, and e_3, and similarly Y_4, X_1, and X_3 form a basis for $W^* = (\mathbb{C}^3)^*$ that corresponds to the dual basis e_1^*, e_2^*, and e_3^*: we have

$$W = \mathbb{C}\{X_4, Y_1, Y_3\}; \qquad W^* = \mathbb{C}\{Y_4, X_1, X_3\};$$

$$\mathfrak{g}_2 = \mathfrak{g}_0 \oplus W \oplus W^*.$$

With these isomorphisms, the brackets

$$\mathfrak{g}_0 \times W \to W \quad \text{and} \quad \mathfrak{g}_0 \times W^* \to W^*$$

correspond to the standard operations of $\mathfrak{sl}_3\mathbb{C}$ on $\mathbb{C}^3$ and $(\mathbb{C}^3)^*$.

Next we look at brackets of elements in W. Note that $[W, W]$ is contained in W^*, either by weights or by looking at the table. The table is

	Y_1	Y_3
X_4	$-2X_3$	$2X_1$
Y_1	0	$-2Y_4$

or

	e_2	e_3
e_1	$-2e_3^*$	$2e_2^*$
e_2	0	$-2e_1^*$

Identifying $W = \mathbb{C}^3$, $W^* = (\mathbb{C}^3)^*$ as above, we see that the bracket $W \times W \to W^*$ becomes the map

$$W \times W \to W^* = \Lambda^2 W, \qquad v \times w \mapsto -2 \cdot v \wedge w.$$

Similarly for W^*, we have $[W^*, W^*] \subset W$, and the bracket is identified with the map

$$W^* \times W^* \to W = \Lambda^2 W^*, \qquad \varphi \times \psi \mapsto 2 \cdot \varphi \wedge \psi.$$

Finally we must look at brackets of elements of W with those of W^*, which land in $\mathfrak{g}_0$. Here the table is

	Y_4	X_1	X_3
X_4	$2H_5 + H_2$	$3X_5$	$3X_6$
Y_1	$3Y_5$	$H_2 - H_5$	$3X_2$
Y_3	$3Y_6$	$3Y_2$	$-H_5 - 2H_2$

In terms of the standard bases, $[e_i, e_j^*] = 3E_{i,j} - \delta_{ij}I$. Intrinsically, this mapping

$$[\ ,\ ]\colon W \times W^* \to \mathfrak{sl}_3\mathbb{C} \subset \mathfrak{gl}(W)$$

can be described by the formula

$$[v, \varphi](w) = 3\varphi(w)v - \varphi(v)w \tag{22.4}$$

for $v, w \in W$ and $\varphi \in W^*$.

Exercise 22.5*. Show that $[v, \varphi]$ is the element of $\mathfrak{sl}_3\mathbb{C}$ characterized by the formula

$$B([v, \varphi], Z) = 18\varphi(Z \cdot v) \quad \text{for all } Z \in \mathfrak{sl}_3\mathbb{C},$$

where B is the Killing form on $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}$. In other words, if we write $v * \varphi$ for the element in $\mathfrak{g}_0 = \mathfrak{sl}_3$ satisfying the identity

$$B(v * \varphi, Z) = \varphi(Z \cdot v) \quad \text{for all } Z \in \mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}, \tag{22.6}$$

then the bracket $[v, \varphi]$ can be written in the form

$$[v, \varphi] = 18 \cdot v * \varphi. \tag{22.7}$$

It is now a relatively painless task to verify the Jacobi identity, since, rather than having to check it for triples from a basis, it suffices to check it on triples of arbitrary elements of the three spaces $\mathfrak{g}_0$, W, and W^* using the above linear algebra descriptions for the brackets. We will write out this exercise, since the same reasoning will be used later. For example, for three or two elements from $\mathfrak{g}_0$, this amounts to the fact that $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}$ is a Lie algebra and W and W^* are representations.

For one element Z in $\mathfrak{g}_0$, and two elements v and w in W, the Jacobi identity for these three elements is equivalent to the identity

$$Z \cdot (v \wedge w) = (Z \cdot v) \wedge w + v \wedge (Z \cdot w),$$

which we know for the action of a Lie algebra on an exterior product; and similarly for one element in $\mathfrak{g}_0$ and two in W^*.

The Jacobi identity for $Z \in \mathfrak{g}_0$, $v \in W$, and $\varphi \in W^*$ amounts to

$$[Z, v * \varphi] = (Z \cdot v) * \varphi + v * (Z \cdot \varphi).$$

Applying $B(Y, —)$ to both sides, and using the identity $B(Y, [Z, X]) = B([Y, Z], X)$, this becomes

$$\varphi([Y, Z] \cdot v) = \varphi(Y \cdot (Z \cdot v)) + (Z \cdot \varphi)(Y \cdot v).$$

Since $\varphi([Y, Z] \cdot v) = \varphi(Y \cdot (Z \cdot v)) - \varphi(Z \cdot (Y \cdot v))$, this reduces to

$$(Z \cdot \varphi)(w) = -\varphi(Z \cdot w),$$

for $w = Y \cdot v$, which comes from the fact that W and W^* are dual representations.

For triples u, v, w in W, the Jacobi identity is similarly reduced to the identity

$$(u \wedge v)(Z \cdot w) + (v \wedge w)(Z \cdot u) + (w \wedge u)(Z \cdot v) = 0$$

for all $z \in \mathfrak{g}_0$, which amounts to

$$\begin{aligned} &u \wedge v \wedge (Z \cdot w) + u \wedge (Z \cdot v) \wedge w + (Z \cdot u) \wedge v \wedge w \\ &= Z \cdot (u \wedge v \wedge w) = 0 \quad \text{in } \textstyle\bigwedge^3 W = \mathbb{C}; \end{aligned}$$

and similarly for triples from W^*.

For $v, w \in W$, and $\varphi \in W^*$, noting that

$$[[v, w], \varphi] = -2 \cdot [v \wedge w, \varphi] = -4 \cdot (v \wedge w) \wedge \varphi = -4 \cdot (\varphi(v)w - \varphi(w)v),$$

the Jacobi identity for these elements reads

$$-4\cdot(\varphi(v)w - \varphi(w)v) = -[w, \varphi](v) + [v, \varphi](w). \tag{22.8}$$

The right-hand side is

$$-[w, \varphi](v) + [v, \varphi](w) = -(3\varphi(v)w - \varphi(w)v) + (3\varphi(w)v - \varphi(v)w),$$

which proves this case. (This last line was the only place where we needed to use the definition (22.4) in place of the fancier (22.7).)

The last case is for one element v in W and two elements φ and ψ in W^*. This time identity to be proved comes down to

$$-4\cdot(\psi(v)\varphi - \varphi(v)\psi) = [v, \varphi]\cdot\psi - [v, \psi]\cdot\varphi.$$

Applying both sides to an element w in W, this becomes

$$-4\cdot(\psi(v)\varphi(w) - \varphi(v)\psi(w)) = \varphi([v, \psi]\cdot w) - \psi([v, \varphi]\cdot w).$$

If we apply ψ to the previous case (22.8) we have

$$-4\cdot(\varphi(v)\psi(w) - \varphi(w)\psi(v)) = -\psi([w, \varphi]\cdot v) + \psi([v, \varphi]\cdot w).$$

And these are the same, using the symmetry of the Killing form:

$$18\cdot\varphi([v, \psi]\cdot w) = B([v, \psi], [w, \varphi]) = B([w, \varphi], [v, \psi]) = 18\psi([w, \varphi]\cdot v).$$

This completes the proof that the algebra with multiplication table (22.1) is a Lie algebra. With the hindsight derived from working all this out, of course, we see that there is a quicker way to construct $\mathfrak{g}_2$, without any multiplication table: simply start with $\mathfrak{sl}_3\mathbb{C} \oplus W \oplus W^*$, and define products according to the above rules.

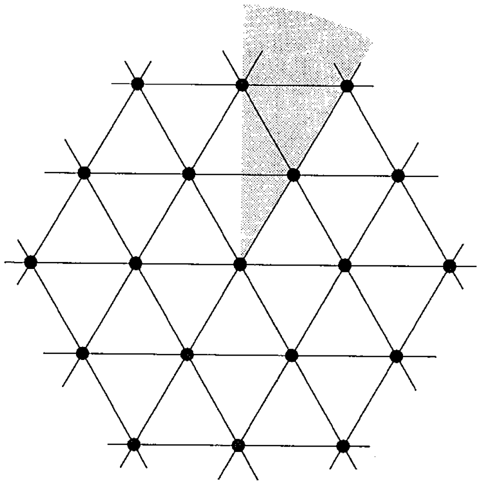
§22.3. Representations of $\mathfrak{g}_2$

We would now like to use the standard procedure, outlined in Lecture 14 (and carried out for the classical Lie algebras in Lectures 15–20) to say something about the representations of $\mathfrak{g}_2$. One nice aspect of this is that, working simply from the root system of $\mathfrak{g}_2$ and analyzing its representations, we will arrive at what is perhaps the simplest description of the algebra: we will see that $\mathfrak{g}_2$ is the algebra of endomorphisms of a seven-dimensional vector space preserving a general trilinear form.

The first step is to find the weight lattice for $\mathfrak{g}_2$. This is the lattice $\Lambda_W \subset \mathfrak{h}^*$ dual to the lattice $\Gamma_W \subset \mathfrak{h}$ generated by the six distinguished elements H_i. By (22.2), Γ_W is generated by H_1 and H_2. Since the values of the eigenvalues α_1 and α_2 on H_1 and H_2 are given by

$$\alpha_1(H_1) = 2, \qquad \alpha_1(H_2) = -1,$$
$$\alpha_2(H_1) = -3, \qquad \alpha_2(H_2) = 2,$$

it follows that the weight lattice is generated by the eigenvalues α_1 and α_2 (and in particular the weight lattice Λ_W is equal to the root lattice Λ_R). The picture is thus

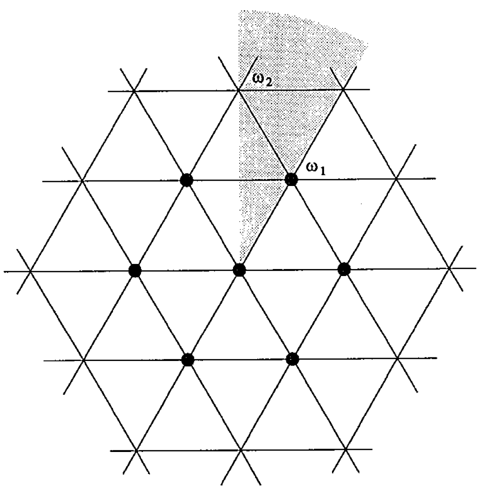


As in the case of the classical Lie algebras, the intersection of the (closed) Weyl chamber $\mathscr{W}$ with the weight lattice is a free semigroup on the two fundamental weights

$$\omega_1 = 2\alpha_1 + \alpha_2 \quad \text{and} \quad \omega_2 = 3\alpha_1 + 2\alpha_2.$$

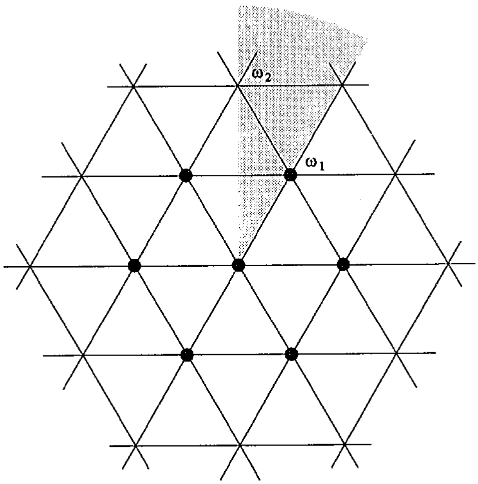
Any irreducible representation of $\mathfrak{g}_2$ will thus have a highest weight vector λ which is a non-negative linear combination of these two. As usual, we write $\Gamma_{a,b}$ for the irreducible representation with highest weight $a\omega_1 + b\omega_2$.

Let us consider first the representation $\Gamma_{1,0}$ with highest weight ω_1. Translating ω_1 around by the action of the Weyl group, we see that the weight diagram of $\Gamma_{1,0}$ looks like



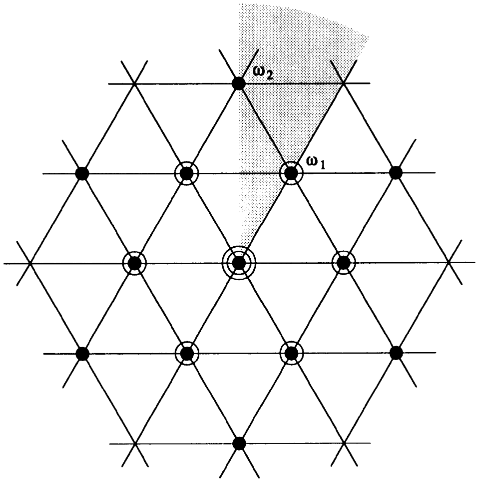
Since there is only one way of getting from the weight ω_1 to the weight 0 by subtraction of simple positive roots, the multiplicity of the weight 0 in $\Gamma_{1,0}$ must be 1. $\Gamma_{1,0}$ is thus a seven-dimensional representation. It is the smallest of the representations of $\mathfrak{g}_2$, and moreover has the property (as we will verify below) that every irreducible representation of $\mathfrak{g}_2$ appears in its tensor algebra; we will therefore call it the *standard* representation of $\mathfrak{g}_2$ and denote it V.

The next smallest representation of $\mathfrak{g}_2$ is the representation $\Gamma_{0,1}$ with highest weight ω_2; this is just the adjoint representation, with weight diagram



Note that the multiplicity of 0 as a weight of $\Gamma_{0,1}$ is 2, and the dimension of $\Gamma_{0,1}$ is 14.

Consider next the exterior square $\wedge^2 V$ of the standard representation $V = \Gamma_{1,0}$ of $\mathfrak{g}_2$. Its weight diagram looks like

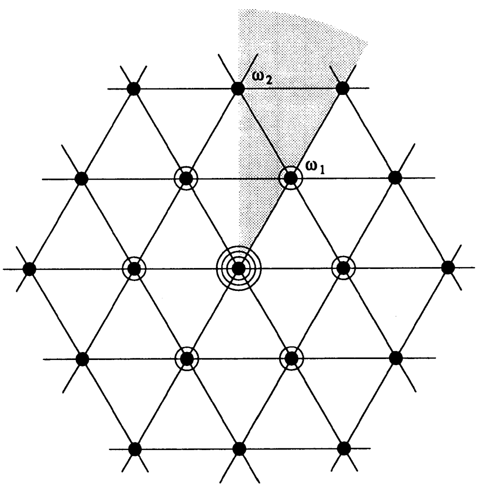


from which we may deduce that

$$\wedge^2 V \cong \Gamma_{0,1} \oplus V.$$

In particular, since the adjoint representation $\Gamma_{0,1}$ of $\mathfrak{g}_2$ is contained in $\wedge^2 V$, and the irreducible representation $\Gamma_{a,b}$ with highest weight $a\omega_1 + b\omega_2$ is contained in the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b \Gamma_{0,1}$, we see that *every irreducible representation of* $\mathfrak{g}_2$ *appears in some tensor power* $V^{\otimes m}$ *of the standard representation*, as stated above.

Next, look at the symmetric square $\mathrm{Sym}^2 V$ of the standard representation. It has weight diagram

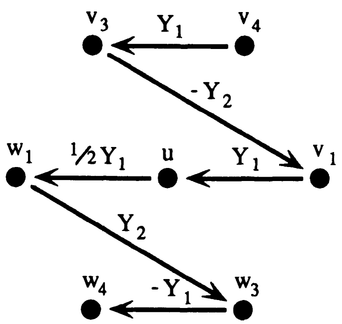


Clearly, this contains a copy of the irreducible representation $\Gamma_{2,0}$ of $\mathfrak{g}_2$ with highest weight $2\omega_1$. Depending on the multiplicities of this representation, it may also contain a copy of V itself, of the trivial representation, or both; or it may be irreducible. To see which is in fact the case, we need to know more about the action of $\mathfrak{g}_2$ on the standard representation V. We will do this in two ways, first by direct calculation, and second using the decomposition of $\mathfrak{g}_2$ into $\mathfrak{sl}_3 \oplus W \oplus W^*$. Although the second approach is shorter, the first illustrates how one can calculate for the exceptional Lie algebras very much as we have been doing in the classical cases.

To describe V explicitly, start with a highest weight vector for V, i.e., any nonzero element v_4 of the eigenspace $V_4 \subset V$ for the action of $\mathfrak{h}$ with eigenvalue α_4. The image of v_4 under the root vector Y_1 will then be a nonzero element of the eigenspace V_3 with eigenvalue α_3 (this follows from the fact that the direct sum $V_3 \oplus V_4$, as a representation of the subalgebra $\mathfrak{s}_{\alpha_1} \subset \mathfrak{g}$, is a copy of the standard representation of $\mathfrak{s}_{\alpha_1} \cong \mathfrak{sl}_2\mathbb{C}$). Similarly, the image of v_3 under Y_2 is a generator v_1 of the eigenspace V_1 with eigenvalue α_1, the image of v_1 under Y_1 is a generator of the eigenspace V_0 with eigenvalue 0, and so on. We may thus choose as a basis for V the vectors

$$v_4, \qquad v_3 = Y_1(v), \qquad v_1 = -Y_2(v_3), \qquad u = Y_1(v_1),$$
$$w_1 = \tfrac{1}{2}Y_1(u), \qquad w_3 = Y_2(w_1), \qquad \text{and} \qquad w_4 = -Y_1(w_3),$$

where v_i (resp. w_i) is an eigenvector with eigenvalue α_i (resp. β_i). (The signs and coefficient $\frac{1}{2}$ in the definition of w_1 are there for reasons of symmetry—see Exercise 22.10.) Diagrammatically, the action of $\mathfrak{g}_2$ may be represented by the arrows



Exercise 22.9. (i) Verify that the vectors v_i, w_i, and u, as defined above, are indeed generators of the corresponding eigenspaces. (ii) Find, in terms of this basis for V, the images of v_4 under the elements Y_3, Y_4, Y_5, and Y_6.

Exercise 22.10. Show that the elements X_i and $Y_i \in \mathfrak{g}_2$ all carry basis vectors v_j and w_j into other basis vectors, up to sign (or to zero, of course), and carry u to twice basis vectors, that is, $X_i u = 2v_i$ and $Y_i u = 2w_i$ for $i = 1, 3, 4$.

Now, the representation $\mathrm{Sym}^2 V$ has, as basis, the pairwise products of the basis vectors for V; and the subrepresentation $\Gamma_{2,0}$ is just the subspace generated by the images of the highest weight vector v_4^2 under (repeated applications of) the generators Y_1, Y_2 of the negative root spaces of $\mathfrak{g}_2$. Thus, for example, the eigenspace in $\mathrm{Sym}^2 V$ with eigenvalue α_4 is the span of the products $u \cdot v_4$ and $v_3 \cdot v_1$; the part of this lying in $\Gamma_{2,0}$ will be the span of the two vectors $Y_2 Y_1 Y_1(v_4^2)$ and $Y_1 Y_2 Y_1(v_4^2)$. We calculate:

$$\begin{aligned} Y_2 Y_1 Y_1(v_4^2) &= Y_2 Y_1(2v_3 \cdot v_4) = Y_2(2v_3^2) \\ &= -4v_1 \cdot v_3 \end{aligned}$$

and

$$\begin{aligned} Y_1 Y_2 Y_1(v_4^2) &= Y_1 Y_2(2v_3 \cdot v_4) = -Y_1(2v_1 \cdot v_4) \\ &= -2v_1 \cdot v_3 - 2u \cdot v_4. \end{aligned}$$

We see, in other words, that $\Gamma_{2,0}$ assumes the weight α_4 with multiplicity 2, so that in particular $\mathrm{Sym}^2 V$ does not contain a copy of V.

Similarly, to see whether or not $\mathrm{Sym}^2 V$ contains a copy of the trivial representation, we have to calculate the multiplicity of the weight 0 in $\Gamma_{2,0}$. Since any path in the weight lattice from the eigenvalue $2\alpha_4$ to 0 obtained by subtracting α_1 and α_2 must pass through α_4, we can do this by evaluating the products of Y_1 and Y_2 on the generators $v_1 \cdot v_3$ and $u \cdot v_4$ of the eigenspace with eigenvalue α_4: we have

$$\begin{aligned} Y_1 Y_1 Y_2(v_1 v_3) &= -Y_1 Y_1(v_1^2) = -Y_1(2u \cdot v_1) \\ &= -4w_1 \cdot v_1 - 2u^2; \\ Y_1 Y_1 Y_2(u \cdot v_4) &= 0; \\ Y_1 Y_2 Y_1(v_1 v_3) &= Y_1 Y_2(u \cdot v_3) = -Y_1(u \cdot v_1) \\ &= -2w_1 \cdot v_1 - u^2; \\ Y_1 Y_2 Y_1(u \cdot v_4) &= Y_1 Y_2(u \cdot v_3 + 2w_1 \cdot v_4) \\ &= Y_1(-u \cdot v_1 + 2w_3 \cdot v_4) \\ &= -2w_1 \cdot v_1 - u^2 - 2w_4 \cdot v_4 + 2w_3 v_3; \\ Y_2 Y_1 Y_1(v_1 v_3) &= Y_2 Y_1(u \cdot v_3) = Y_2(2w_1 \cdot v_3) \\ &= -2w_1 \cdot v_1 + 2w_3 \cdot v_3; \end{aligned}$$

and

$$\begin{aligned} Y_2 Y_1 Y_1(u \cdot v_4) &= Y_2 Y_1(u \cdot v_3 + 2w_1 \cdot v_4) = Y_2(4w_1 \cdot v_3) \\ &= -4w_1 \cdot v_1 + 4w_3 \cdot v_3. \end{aligned}$$

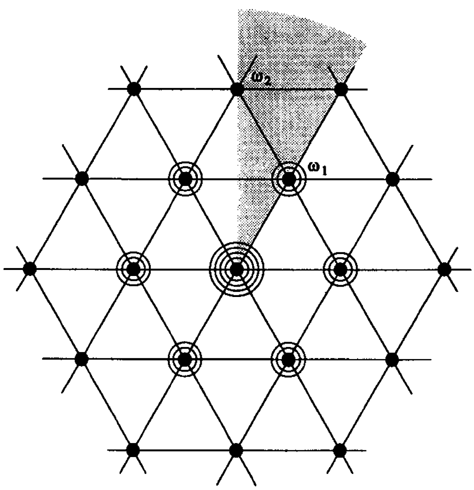
We see from this that the 0-eigenspace of $\Gamma_{2,0}$ is three dimensional; we thus have the decomposition

$$\mathrm{Sym}^2 V \cong \Gamma_{2,0} \oplus \mathbb{C}.$$

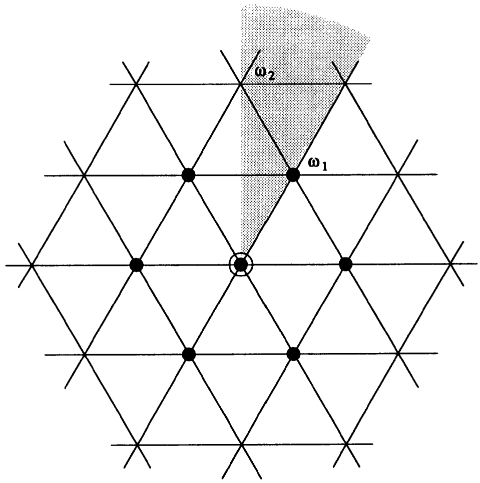
In particular, we deduce that *the action of* $\mathfrak{g}_2$ *on the standard representation* $V = \mathbb{C}^7$ *preserves a quadratic form*; and correspondingly that the subalgebra $\mathfrak{g}_2 \subset \mathfrak{sl}(V) = \mathfrak{sl}_7\mathbb{C}$ is actually contained in the algebra $\mathfrak{so}_7\mathbb{C}$. We will see this again in the following section, where we will give alternative descriptions of the exceptional Lie algebras, and again in §23.3 where we describe compact homogeneous spaces for Lie groups.

Exercise 22.11. Analyze in general the symmetric powers $\mathrm{Sym}^k V$ of the standard representation V of $\mathfrak{g}_2$.

Finally, consider the exterior cube $\Lambda^3 V$ of the standard representation. The weight diagram is



and after we remove one copy of the representation $\Gamma_{2,0}$ with highest weight $2\omega_1$ (this is the sum of the three highest weights α_4, α_3, and α_1 of V), we are left with



This, by what we have seen, can only be the direct sum of the standard representation V with the trivial representation $\mathbb{C}$. In sum, then, we conclude that

$$\bigwedge^3 V \cong \Gamma_{2,0} \oplus V \oplus \mathbb{C}.$$

Note in particular that, as a corollary, *the action of $\mathfrak{g}_2$ on the standard representation preserves a skew-symmetric trilinear form ω on V.* It is not hard to write down this form: it is a linear combination of the five vectors $w_3 \wedge u \wedge v_3, v_4 \wedge u \wedge w_4, w_1 \wedge u \wedge v_1, v_1 \wedge v_3 \wedge w_4$, and $w_1 \wedge w_3 \wedge v_4$; and the fact that it is preserved by X_1 and X_2 is enough to determine the coefficients: we have

$$\omega = w_3 \wedge u \wedge v_3 + v_4 \wedge u \wedge w_4 + w_1 \wedge u \wedge v_1$$
$$+ 2v_1 \wedge v_3 \wedge w_4 + 2w_1 \wedge w_3 \wedge v_4.$$

The fact that the action of $\mathfrak{g}_2$ on V preserves the skew-symmetric cubic form ω takes on additional significance when we make a naive dimension count. The space $\bigwedge^3 V$ of all such alternating forms has dimension 35, while the algebra $\mathfrak{gl}(V)$ of endomorphisms of V has dimension 49; the difference is exactly the dimension of the algebra $\mathfrak{g}_2$. In fact, we can check directly that the linear map

$$\varphi\colon \mathfrak{gl}(V) \to \bigwedge^3 V$$

sending $A \in \operatorname{End}(V)$ to $A(\omega)$ is surjective. We deduce that ω is a general cubic alternating form [i.e., an open dense subset of $\bigwedge^3 V$ corresponds to forms equivalent to ω under $\operatorname{Aut}(V)$], and hence that

Proposition 22.12. *The algebra $\mathfrak{g}_2$ is exactly the algebra of endomorphisms of a seven-dimensional vector space V preserving a general skew-symmetric cubic form ω on V.*

Exercise 22.13*. Verify that the map φ above is surjective by direct calculation of the action of $\mathfrak{gl}(V)$ on $\omega \in \wedge^3 V$.

Exercise 22.14. As an alternative to the preceding exercise, analyze skew-symmetric trilinear forms on $\mathbb{C}^n$ to show that for $n \leq 7$ there are only finitely many such forms, up to the action of $\mathrm{GL}_n\mathbb{C}$. Verify that the form ω above is general in $\wedge^3\mathbb{C}^7$. (In fact, there are only finitely many cubic alternating forms on $\mathbb{C}^8$ as well, though this is fairly complicated; for $n \geq 9$ a simple dimension count shows that there is a continuously varying family of such forms.)

Note that the cubic form ω preserved by the action of $\mathfrak{g}_2$ gives us explicitly the inclusion

$$V \hookrightarrow \wedge^2 V$$

deduced earlier from their weight diagrams: this is just the map $V^* \to \wedge^2 V$ given by contraction/wedge product with ω, composed with the isomorphism of V with V^*.

Exercise 22.15*. Find the algebra of endomorphisms of a six-dimensional vector space preserving a general skew-symmetric trilinear form.

We will see the form ω again when we describe $\mathfrak{g}_2$ in the following section.

These calculations using the table amount to using all the information that can be extracted from the subalgebras $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$ of $\mathfrak{g}_2$. Using the copy of $\mathfrak{sl}_3\mathbb{C}$ that we found in the second section can make some of this more transparent. Make the identification

$$\mathfrak{g}_2 = \mathfrak{g}_0 \oplus W \oplus W^* = \mathfrak{sl}_3\mathbb{C} \oplus W \oplus W^*.$$

As a representation of $\mathfrak{sl}_3\mathbb{C}$, the seven-dimensional representation V must be the sum of W, W^*, and the trivial representation $\mathbb{C}$. If we make this identification,

$$V = W \oplus W^* \oplus \mathbb{C},$$

it is not hard to work out how the rest of $\mathfrak{g}_2$ acts. This is given in the following table:

		W	W^*	$\mathbb{C}$
		w	ψ	z
$\mathfrak{g}_0$	X	$X \cdot w$	$X \cdot \psi$	0
W	v	$-v \wedge w$	$\psi(v)$	$2z \cdot v$
W^*	φ	$\varphi(w)$	$\varphi \wedge \psi$	$2z \cdot \varphi$

With this identification, we have $u = 1$ in $\mathbb{C}$, and

$$v_4 = e_1, \qquad w_1 = e_2, \qquad w_3 = e_3 \quad \text{in } W = \mathbb{C}^3;$$

$$w_4 = e_1^*, \qquad v_1 = e_2^*, \qquad v_3 = e_3^* \quad \text{in } W^* = (\mathbb{C}^3)^*.$$

Conversely, it is not hard to verify that the above table defines a representation of $\mathfrak{g}_2$, by checking the various cases of the identity $[\xi, \eta]\cdot y = \xi\cdot(\eta\cdot y) - \eta\cdot(\xi\cdot y)$ for ξ, η in $\mathfrak{g}_2$ and y in V. Note that the cubic form ω becomes

$$\omega = \sum_{i=1}^{3} e_i \wedge u \wedge e_i^* + 2(e_1 \wedge e_2 \wedge e_3 + e_1^* \wedge e_2^* \wedge e_3^*).$$

This description of V can be used to verify the calculations made earlier, and also to study its symmetric and exterior powers. For example, $\mathrm{Sym}^2 V$ decomposes over $\mathfrak{sl}_3\mathbb{C}$ into

$$\begin{aligned}&\mathrm{Sym}^2 W \oplus \mathrm{Sym}^2 W^* \oplus \mathrm{Sym}^2\mathbb{C} \oplus W \otimes \mathbb{C} \oplus W^* \otimes \mathbb{C} \oplus W \otimes W^* \\ &= \mathrm{Sym}^2 W \oplus \mathrm{Sym}^2 W^* \oplus \mathbb{C} \oplus W \oplus W^* \oplus \mathfrak{sl}_3\mathbb{C} \oplus \mathbb{C}.\end{aligned}$$

To get the weights around the outside ring, the irreducible representation $\Gamma_{2,0}$ must include $\mathrm{Sym}^2 W$, $\mathrm{Sym}^2 W^*$, and $\mathfrak{sl}_3\mathbb{C}$. Checking that $W \subset \mathfrak{g}_2$ maps $\mathrm{Sym}^2 W^*$ nontrivially to W^* shows that it must also include W and W^*. To finish it suffices to compute the part killed by $\mathfrak{g}_2$, which must lie in the sum of the two components which are trivial for $\mathfrak{sl}_3\mathbb{C}$; checking that this is one dimensional, one recovers the decomposition

$$\mathrm{Sym}^2 V = \Gamma_{2,0} \oplus \mathbb{C}.$$

Exercise 22.16. Use this method to decompose $\wedge^3 V$ and $\mathrm{Sym}^3 V$.

§22.4. Algebraic Constructions of the Exceptional Lie Algebras

In this section we will sketch a few of the abstract approaches to the construction of the five exceptional Lie algebras. The constructions are not as easy as you might wish: although the exceptional Lie groups and their Lie algebras have a remarkable way of showing up unexpectedly in many areas of mathematics and physics, they do not have such simple descriptions as the classical series. Indeed, they were not discovered until the classification theorem forced mathematicians to look for them.

To begin with, the method we used to construct $\mathfrak{g}_2$ in the second section of this lecture can be generalized to construct other Lie algebras. This is the construction of Freudenthal, which we do first. It can be used to construct the Lie algebra $\mathfrak{e}_8$ for the diagram (E_8). From $\mathfrak{e}_8$ it is possible to construct $\mathfrak{e}_7$ and $\mathfrak{e}_6$ and $\mathfrak{f}_4$. Then we will present (or at least sketch) several other approaches to their construction. Since it is a rather technical subject, probably not really suited for a first course, we will touch on several approaches rather than give a detailed discussion of one.

The construction of $\mathfrak{g}_2$ as a sum $\mathfrak{g}_0 \oplus W \oplus W^*$ that we found in the second section works more generally, with very little change. Suppose $\mathfrak{g}_0$ is a semisimple Lie algebra, and W is a representation of $\mathfrak{g}_0$; let W^* be the dual representation, and set

$$\mathfrak{g} = \mathfrak{g}_0 \oplus W \oplus W^*.$$

We also need maps

$$\wedge : \wedge^2 W \to W^* \quad \text{and} \quad \wedge : \wedge^2 W^* \to W$$

of representations of $\mathfrak{g}_0$. We assume these are given by trilinear maps of $\mathfrak{g}_0$-representations $T\colon \wedge^3 W \to \mathbb{C}$ and $T'\colon \wedge^3 W^* \to \mathbb{C}$, which means that

$$(u \wedge v)(w) = T(u, v, w) \quad \text{and} \quad \vartheta(\varphi \wedge \psi) = T'(\varphi, \psi, \vartheta).$$

We can then define a bracket on $\mathfrak{g}$ by the same rules as in the second section. To describe it, we let $X, Y, Z, \dots$ denote arbitrary elements of $\mathfrak{g}_0$, $u, v, w, \dots$ elements of W, and $\varphi, \psi, \vartheta, \dots$ elements of W^*. The bracket in $\mathfrak{g}$ is determined by setting:

(i) $[X, Y] = [X, Y]$ (the given bracket in $\mathfrak{g}_0$),
(ii) $[X, v] = X \cdot v$ (the action of $\mathfrak{g}_0$ on W),
(iii) $[X, \varphi] = X \cdot \varphi$ (the canonical action of $\mathfrak{g}_0$ on W^*),
(iv) $[v, w] = a \cdot (v \wedge w)$ (for a scalar a to be determined),
(v) $[\varphi, \psi] = b \cdot (\varphi \wedge \psi)$ (for a scalar b to be determined)
(vi) $[v, \varphi] = c \cdot (v * \varphi)$ (for a scalar c to be determined).

As before, $v * \varphi$ is the element of $\mathfrak{g}_0$ such that

$$B(v * \varphi, Z) = \varphi(Z \cdot v) \quad \text{for all } Z \in \mathfrak{g}_0,$$

where B is the Killing form on $\mathfrak{g}_0$. The rules (i)–(vi) determine a bilinear product [,] on all of $\mathfrak{g}$, and the fact that it is skew follows from the facts that $[X, X] = 0$, $[v, v] = 0$, and $[\varphi, \varphi] = 0$.

The argument that we gave showing that $\mathfrak{g}_2$ satisfies the Jacobi identity works in this general case without essential change, except for the last two cases, where explicit calculation is needed. For $v, w \in W$, and $\varphi \in W^*$, the Jacobi identity is equivalent to the identity

$$ab((v \wedge w) \wedge \varphi) = c((v * \varphi) \cdot w - (w * \varphi) \cdot v). \tag{22.17}$$

For $v \in W$, $\varphi, \psi \in W^*$, the Jacobi identity amounts to

$$ab((\varphi \wedge \psi) \wedge v) = c((v * \psi) \cdot \varphi - (v * \varphi) \cdot \psi). \tag{22.18}$$

We will see in Exercise 22.20 that (22.17) and (22.18) are equivalent. Again, the simplicity of the resulting Lie algebra is easy to see, provided all the weight spaces are one dimensional, using Exercise 14.34, so we have:

Proposition 22.19 (Freudenthal). *Given a representation W of a semisimple Lie algebra $\mathfrak{g}_0$ and trilinear forms T and T' inducing maps $\bigwedge^2 W \to W^*$ and $\bigwedge^2 W^* \to W$, such that (22.17) and (22.18) are satisfied, the above products make*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus W \oplus W^*$$

into a Lie algebra. If the weight spaces of W are all one dimensional, and the weights of W, W^*, and the roots of $\mathfrak{g}_0$ are all distinct, and $abc \neq 0$, then $\mathfrak{g}$ is semisimple, with the same Cartan subalgebra as $\mathfrak{g}_0$.

Exercise 22.20*. (a) Show that the trilinear map T determines a map $\wedge$: $\bigwedge^2 W \to W^*$ of representations if and only if it satisfies the identity

$$T(X \cdot u, v, w) + T(u, X \cdot v, w) + T(u, v, X \cdot w) = 0 \qquad \forall X \in \mathfrak{g}_0,$$

and similarly for T'.

(b) Show that each of (22.17) and (22.18) is equivalent to the identity

$$ab \cdot (v \wedge w)(\varphi \wedge \psi) = c \cdot (B(w * \psi, v * \varphi) - B(w * \varphi, v * \psi)).$$

The Lie algebra $\mathfrak{e}_8$ for (E_8) can be constructed by this method. This time $\mathfrak{g}_0$ is taken to be the Lie algebra $\mathfrak{sl}_9\mathbb{C}$; if $V = \mathbb{C}^9$ is the standard representation of $\mathfrak{sl}_9\mathbb{C}$, let $W = \bigwedge^3 V$, so $W^* = \bigwedge^3 V^*$; the trilinear map is the usual wedge product

$$\bigwedge^3 V \otimes \bigwedge^3 V \otimes \bigwedge^3 V \to \bigwedge^9 V = \mathbb{C},$$

and similarly for $\bigwedge^3 V^*$. We leave the verifications to the reader:

Exercise 22.21*. (i) Verify the conditions on the roots of $\mathfrak{sl}_9$ and the weights of $\bigwedge^3 V$ and $\bigwedge^3 V^*$. (ii) Use the fact that $B(X, Y) = 18 \cdot \mathrm{Tr}(XY)$ for $\mathfrak{sl}_9$ to show that (22.17) holds precisely if $c = -18ab$. (iii) Show that the Dynkin diagram of the resulting Lie algebra is (E_8).

Note that the dimension of $\mathfrak{sl}_9\mathbb{C}$ is 80, and that of W and W^* is 84, so the sum has dimension 248, as predicted by the root system of (E_8).

Once the Lie algebra $\mathfrak{e}_8$ is constructed, $\mathfrak{e}_7$ and $\mathfrak{e}_6$ can be found as subalgebras, as follows. Note that removing one or two nodes from the long arm of the Dynkin diagram of (E_8) leads to the Dynkin diagrams (E_7) and (E_6).

In general, if $\mathfrak{g}$ is a simple Lie algebra, with Dynkin diagram D, consider a subdiagram D° of D obtained by removing some subset of nodes, together with all the lines meeting these nodes.[1] Then we can construct a semisimple subalgebra $\mathfrak{g}^\circ$ of $\mathfrak{g}$ with D° as its Dynkin diagram. In fact, $\mathfrak{g}^\circ$ is the subalgebra generated by all the root spaces $\mathfrak{g}_{\pm\alpha}$, where α is a root in D°.

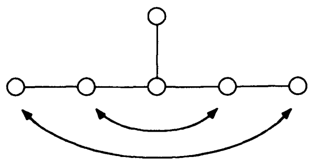
[1] If there are double or triple lines between two nodes, both nodes should be removed or kept together.

Exercise 22.22. (a) Prove this by verifying that the positive roots of $\mathfrak{g}^\circ$ are the positive roots β of $\mathfrak{g}$ that are sums of the roots in D°, and the Cartan subalgebra $\mathfrak{h}^\circ$ is spanned by the corresponding vectors $H_\beta \in \mathfrak{h}$.

(b) Carry this out for $\mathfrak{e}_7$ and $\mathfrak{e}_6$; in particular, show again that $\mathfrak{e}_7$ has 63 positive roots, so dimension $7 + 2(63) = 133$, and $\mathfrak{e}_6$ has 36 positive roots, so dimension $6 + 2(36) = 78$.

Exercise 22.23. For each of the simple Lie algebras, find the subalgebras obtained by removing one node from an end of its Dynkin diagram.

The last exceptional Lie algebra $\mathfrak{f}_4$ can be constructed by taking an invariant subalgebra of $\mathfrak{e}_6$ by an involution. This involution corresponds to the evident symmetry in the Dynkin diagram:



In general, an automorphism of a Dynkin diagram arises from an automorphism of the corresponding semisimple Lie algebra, as follows from the fact that the multiplication table is determined by the Dynkin diagram, cf. Proposition 21.22 and Claim 21.25.

Exercise 22.24*. (a) Show that the invariant subalgebra for the indicated involution of $\mathfrak{e}_6$ is a simple Lie algebra $\mathfrak{f}_4$ with Dynkin diagram (F_4).

(b) Find the invariant subalgebra for the involutions of (A_n) and (D_n), and for an automorphism of order three of (D_4).

Exercise 22.25*. For each automorphism of the Dynkin diagrams (A_n) and (D_n), find an explicit automorphism of $\mathfrak{sl}_{n+1}\mathbb{C}$ and $\mathfrak{so}_{2n}\mathbb{C}$ that induces it.

The exceptional Lie algebras can also be realized as the Lie algebras of derivations of certain nonassociative algebras. This also gives realizations of corresponding Lie groups as groups of automorphism of these algebras (see Exercise 8.28). Some examples of this for associative algebras should be familiar. The group of automorphisms of the algebra $\mathbb{H}$ of (real) quaternions is $O(3)$, so the Lie algebra of derivations is $\mathfrak{so}_3\mathbb{R}$. The Lie algebra of derivations of the complexification $\mathbb{H}_\mathbb{C}$ is $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$.

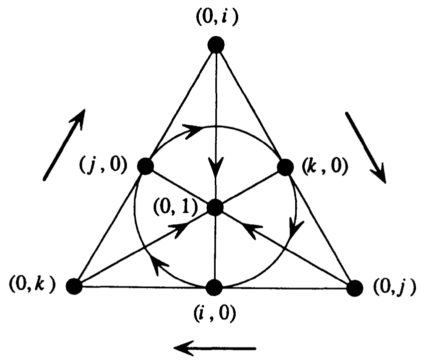
The exceptional group G_2 can be realized as the group of automorphisms of the complexification of the eight-dimensional *Cayley algebra*, or algebra of *octonions*. Recall that the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ can be constructed as the set of pairs (a, b) of complex numbers. In a similar way the Cayley algebra, which we denote by $\mathbb{O}$, can be constructed as the set of pairs (a, b), with a and b quaternions. The addition is componentwise, with multiplication

$$(a, b) \circ (c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

where $^-$ denotes conjugation in $\mathbb{H}$. This algebra $\mathbb{O}$ also has a conjugation, which takes (a, b) to $(\bar{a}, -b)$. It has a basis $1 = (1, 0)$, together with seven elements $e_1, \ldots, e_7$:

$$(i, 0), (j, 0), (k, 0), (0, 1), (0, i), (0, j), (0, k).$$

These satisfy $e_p \circ e_p = -1$ and $e_p \circ e_q = -e_q \circ e_p$ for $p \neq q$, and the conjugate $\overline{e_p}$ of e_p is $-e_p$. The multiplication table can be encoded in the diagram:



Here, if e_p, e_q, and e_r appear on a line in the order shown by the arrow, then

$$e_p \circ e_q = e_r, \qquad e_q \circ e_r = e_p, \qquad e_q \circ e_p = e_r.$$

Note in particular that any two of these basic elements generate a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

Exercise 22.26. Show that the subalgebra of $\mathbb{O}$ generated by any two elements is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Deduce that, although $\mathbb{O}$ is noncommutative and nonassociative, it is "alternative," i.e., it satisfies the identities $(x \circ x) \circ y = x \circ (x \circ y)$ and $y \circ (x \circ x) = (y \circ x) \circ x$.

A trace and norm can be defined on $\mathbb{O}$ by

$$\mathrm{Tr}(x) = \tfrac{1}{2}(x + \bar{x}), \qquad N(x) = x \circ \bar{x};$$

these satisfy the relation $x^2 - 2\,\mathrm{Tr}(x) + N(x) = 0$. Let $\beta(x, y) = \frac{1}{2}(x \circ \bar{y} + y \circ \bar{x})$ be the bilinear form associated to N; note that the above basis is an orthonormal basis for this inner product.

Let G be the group of algebra automorphisms of the real algebra $\mathbb{O}$. The next exercise sketches a proof that the complexification of G is a Lie group of type (G_2).

Exercise 22.27*. The center of $\mathbb{O}$ is $\mathbb{R}\cdot 1$, which is preserved by G. Let Y be orthogonal space to $\mathbb{R}\cdot 1$ with respect to the quadratic form N. Then G is imbedded in the group $SO(Y)$ of orthogonal transformations of Y.

(a) Define a "cross product" $\times$ on Y by the formula $v \times w = v\cdot w + \beta(v, w)\cdot 1$. Show that G can be identified with the group of orthogonal transformations of Y that preserve the cross product.
(b) Show that $G = \mathrm{Aut}(\mathbb{O})$ acts transitively on the 6-sphere

$$S^6 = \{\textstyle\sum r_i e_i \colon \sum r_i^2 = 1\},$$

and the subgroup K that fixes $i = e_1$ is mapped onto the 5-sphere in $e_1^\perp$ by the map $g \mapsto g\cdot j$. Conclude from this that G is 14-dimensional and simply connected.
(c) Show that $\{D \in \mathrm{Der}(\mathbb{O})\colon D(i) = 0\}$ is isomorphic to $\mathfrak{su}_3$.
(d) Verify that the Lie algebra of derivations of the complex octonians is the simple Lie algebra of type (G_2).

Exercise 22.28*. The octonions can also be constructed from the Clifford algebra of an eight-dimensional vector space with a nondegenerate quadratic form. With V, S^+, and S^- as in §20.3, with $v_1 \in V$, $s_1 \in S^+$, $t_1 = v_1\cdot s_1 \in S^-$ chosen so the values of the quadratic forms are 1 on each of them as in Exercise 20.50, define a product $V \times V \to V$, $(v, w) \mapsto v \circ w$ by the formula

$$v \circ w = (v\cdot t_1)\cdot(w\cdot s_1).$$

Note that $v\cdot t_1 \in S^+$, $w\cdot s_1 \in S^-$, so their product $(v\cdot t_1)\cdot(w\cdot s_1)$ is back in V.

(a) Show that V with this product is isomorphic to the complex octonians $\mathbb{O}$, with unit v_1, with the map $v \mapsto -\rho(v_1)(v)$ corresponding to conjugation in $\mathbb{O}$.

Conversely, starting with the complex octonians $\mathbb{O}$, one can reconstruct the algebra of §20.3: define $A = \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$, define an automorphism J of order 3 of A by $J(x, y, z) = (z, x, y)$, and define a product $\cdot$ from each succession of two factors to the third by the formulas $x\cdot y = \bar{x} \circ \bar{y}$, $y\cdot z = \bar{y} \circ \bar{z}$, $z\cdot x = \bar{z} \circ \bar{x}$.

(b) Show that A is isomorphic to the algebra described in §20.3.

(c) Identifying $\mathfrak{so}_8\mathbb{C}$ with the space of skew linear transformations of $\mathbb{O}$, show that for each A in $\mathfrak{so}_8\mathbb{C}$ there are unique B and C in $\mathfrak{so}_8\mathbb{C}$ such that

$$A(x \circ y) = B(x) \circ y + x \circ C(y)$$

for all complex octonions x and y. Equivalently, if one defines a trilinear form $(\ ,\ ,\ )$ on the octonions by $(x, y, z) = \mathrm{Tr}((x \circ y) \circ z) = \mathrm{Tr}(x \circ (y \circ z))$,

$$(Ax, y, z) + (x, By, z) + (x, y, Cz) = 0$$

for all x, y, z. Show that this trilinear form agrees with that defined in Exercise 20.49, and the mapping $A \mapsto B$ determines the triality automorphism j' of $\mathfrak{so}_8\mathbb{C}$ of order three described in Exercise 20.51.

Exercise 22.29. Define three homomorphisms from the real Clifford algebra $C_7 = C(0, 7)$ to $\mathrm{End}_{\mathbb{R}}(\mathbb{O})$ by sending $v \in \mathbb{R}^7 = \sum \mathbb{R}e_i$ to the maps L_v, R_v, and T_v defined by $L_v(x) = v \circ x$, $R_v(x) = x \circ v$, and $T_v(x) = v \circ (x \circ v) = (v \circ x) \circ v$.

(a) Show that these do determine maps of the Clifford algebra, and that the induced maps

$$\mathrm{Spin}_8 \mathbb{R} \hookrightarrow C_8^{\mathrm{even}} = C_7 \to \mathrm{End}_{\mathbb{R}}(\mathbb{O})$$

are the two spin representations and the standard representation, respectively.

(b) Verify that $T_v(x \circ y) = L_v(x) \cdot L_v(y)$ for all v, x, y, and use this to verify the triality formula in (c) of the preceding exercise.

The algebra $\mathfrak{f}_4$ can be realized as the derivation algebra of the complexification of a 27-dimensional *Jordan algebra* $\mathbb{J}$. This can be constructed as the set of matrices of the form

$$\begin{pmatrix} a & \alpha & \beta \\ \bar{\alpha} & b & \gamma \\ \bar{\beta} & \bar{\gamma} & c \end{pmatrix},$$

with a, b, c scalars, and α, β, γ in $\mathbb{O}$. The product $\circ$ in $\mathbb{J}$ is given by

$$x \circ y = \tfrac{1}{2}(xy + yx),$$

where the products on the right-hand side are defined by usual matrix multiplication. This algebra is commutative but not associative, and satisfies the identity $((x \circ x) \circ y) \circ x = (x \circ x) \circ (y \circ x)$. In fact, (F_4) is the group of automorphisms of this 27-dimensional space that preserve the scalar product $(x, y) = \mathrm{Tr}(x \circ y)$ and the scalar triple product $(x, y, z) = \mathrm{Tr}((x \circ y) \circ z)$. The kernel of the trace map is an irreducible 26-dimensional representation of $\mathfrak{f}_4$. For details see [Ch-S], [To], [Pos].

In addition, there is a cubic form "det" on $\mathbb{J}$ such that the linear automorphisms of $\mathbb{J}$ that preserve this form is a group of type (E_6). This again shows $\mathfrak{f}_4$ as a subalgebra of $\mathfrak{e}_6$.

The other exceptional Lie algebras can also be constructed as derivations of appropriate algebras. We refer for this to [Ti2], [Dr], [Fr2], [Jac2], and the references found in these sources. Other constructions were given by Witt, cf. [Wa]. The simple Lie algebras are also constructed explicitly in [S-K, §1]. See also [Ch-S], [Fr1], and [Sc].

What little we will have to say about the representations of the four exceptional Lie algebras besides $\mathfrak{g}_2$ can wait until we have the Weyl character formula.

LECTURE 23

Complex Lie Groups; Characters

This lecture serves two functions. First and foremost, we make the transition back from Lie algebras to Lie groups: in §23.1 we classify the groups having a given semisimple Lie algebra, and say which representations of the Lie algebra, as described in the preceding lectures, lift to which groups. Secondly, we introduce in §23.2 the notion of *character* in the context of Lie theory; this gives us another way of describing the representations of the classical groups, and also provides a necessary framework for the results of the following two lectures. Then in §23.3 we sketch the beautiful interrelationships among Dynkin diagrams, compact homogeneous spaces and the irreducible representations of a Lie group. The first two sections are elementary modulo a little topology needed to calculate the fundamental groups of the classical groups in §23.1. The third section, by contrast, may appear impossible: it involves, at various points, projective algebraic geometry, holomorphic line bundles, and their cohomology. In fact, a good deal of §23.3 can be understood without these notions; the reader is encouraged to read as much of the section as seems intelligible. A final section §23.4 gives a very brief introduction to the related Bruhat decomposition, which is included because of its ubiquity in the literature.

§23.1. Representations of Complex Simple Lie Groups

In Lecture 21 we classified all simple Lie algebras over $\mathbb{C}$. This in turn yields a classification of simple complex Lie groups: as we saw in Lecture 7, for any Lie algebra $\mathfrak{g}$ there is a unique simply connected group G, and all other (connected) complex Lie groups with Lie algebra $\mathfrak{g}$ are quotients of G by discrete subgroups of the center $Z(G)$. In this section, we will first describe the groups associated to the classical Lie algebras, and then proceed to describe which of the representations of the classical algebras we have described in Part III lift to which of the groups. We start with

Proposition 23.1. *For all $n \geq 1$, the Lie groups $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$ are connected and simply connected. For $n \geq 1$, $\mathrm{SO}_n\mathbb{C}$ is connected, with $\pi_1(\mathrm{SO}_2\mathbb{C}) = \mathbb{Z}$, and $\pi_1(\mathrm{SO}_n\mathbb{C}) = \mathbb{Z}/2$ for $n \geq 3$.*

PROOF. The main tool needed from topology is the long exact homotopy sequence of a fibration. If the Lie group G acts transitively on a manifold M, and H is the isotropy group of a point P_0 of M, then $G/H = M$, and the map $G \to M$ by $g \mapsto g \cdot P_0$ is a fibration with fiber H. The resulting long exact sequence is, assuming the spaces are connected,

$$\cdots \to \pi_2(M) \to \pi_1(H) \to \pi_1(G) \to \pi_1(M) \to \{1\}. \tag{23.2}$$

(The base points, which are omitted in this notation, can be taken to be the identity elements of H and G, and the point P_0 in M.) In practice we will know M and H are connected, from which it follows that G is also connected. From this exact sequence, if M and H are also simply connected, the same follows for G.

To apply the long exact homotopy sequence in our present circumstance we argue by induction, noting first that $\mathrm{SL}_1\mathbb{C} = \mathrm{SO}_1\mathbb{C} = \{1\}$. Now consider the action of $G = \mathrm{SL}_n\mathbb{C}$ on the manifold $M = \mathbb{C}^n \backslash \{0\}$. The subgroup H fixing the vector $P_0 = (1, 0, \ldots, 0)$ consists of matrices whose first column is $(1, 0, \ldots, 0)$ and whose lower right $(n-1)$ by $(n-1)$ matrix is in $\mathrm{SL}_{n-1}\mathbb{C}$; it follows that as topological spaces $H \cong \mathrm{SL}_{n-1}\mathbb{C} \times \mathbb{C}^{n-1}$. Since M is simply connected for $n \geq 2$ (having the sphere S^{2n-1} as a deformation retract), and H has $\mathrm{SL}_{n-1}\mathbb{C}$ as a deformation retract, the claim for $\mathrm{SL}_n\mathbb{C}$ follows from (23.2) by induction on n.

The group $\mathrm{SO}_2\mathbb{C}$ is isomorphic to the multiplicative group $\mathbb{C}^*$, which has the circle as a deformation retract, so $\pi_1(\mathrm{SO}_2\mathbb{C}) = \mathbb{Z}$. The group $G = \mathrm{SO}_n\mathbb{C}$ acts transitively on $M = \{v \in \mathbb{C}^n: Q(v, v) = 1\}$, where Q is the symmetric bilinear form preserved by G. (The transitivity of the action is more or less equivalent to knowing that all nondegenerate symmetric bilinear forms are equivalent.) For explicit calculations take the standard Q for which the standard basis $\{e_i\}$ of $\mathbb{C}^n$ is an orthonormal basis. This time the subgroup H fixing e_1 is $\mathrm{SO}_{n-1}\mathbb{C}$. From the following exercise, it follows that M has the sphere S^{n-1} as a deformation retract. By (23.2) the map

$$\pi_1(\mathrm{SO}_{n-1}\mathbb{C}) \to \pi_1(\mathrm{SO}_n\mathbb{C})$$

is an isomorphism for $n \geq 4$. So it suffices to look at $\mathrm{SO}_3\mathbb{C}$. This could be done by looking at the maps in the sme exact sequence, but we saw in Lecture 10 that $\mathrm{SO}_3\mathbb{C}$ has a two-sheeted covering by $\mathrm{SL}_2\mathbb{C}$, which is simply connected by the preceding paragraph, so $\pi_1(\mathrm{SO}_3\mathbb{C}) = \mathbb{Z}/2$, as required.

The group $G = \mathrm{Sp}_{2n}\mathbb{C}$ acts transitively on

$$M = \{(v, w) \in \mathbb{C}^{2n} \times \mathbb{C}^{2n}: Q(v, w) = 1\},$$

where Q is the skew form preserved by G, and the isotropy group is $\mathrm{Sp}_{2n-2}\mathbb{C}$. Since $\mathrm{Sp}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}$, the first case is known. By the following exercise, since M is defined in $\mathbb{C}^{4n}$ by a nondegenerate quadratic form, M has S^{4n-1} as a deformation retract, so we conclude again by induction. □

Exercise 23.3*. Show that $\{(z_1, \ldots, z_n) \in \mathbb{C}^n: \sum z_i^2 = 1\}$ is homeomorphic to the tangent bundle to the $(n-1)$-sphere, i.e., to

$$T_{S^{n-1}} = \{(u, v) \in S^{n-1} \times \mathbb{R}^n: u \cdot v = 0\}.$$

Using the exact sequence $\{1\} \to \mathrm{SL}_n\mathbb{C} \to \mathrm{GL}_n\mathbb{C} \to \mathbb{C}^* \to \{1\}$ we deduce from the proposition and (23.2) that

$$\pi_1(\mathrm{GL}_n\mathbb{C}) = \mathbb{Z}. \tag{23.4}$$

Exercise 23.5. Show that for all the above groups G, the second homotopy groups $\pi_2(G)$ are trivial.

We digress a moment here to mention a famous fact. Each of the above groups G has an associated compact subgroup: $\mathrm{SU}(n) \subset \mathrm{SL}_n\mathbb{C}$, $\mathrm{Sp}(n) \subset \mathrm{Sp}_{2n}\mathbb{C}$, and $\mathrm{SO}(n) \subset \mathrm{SO}_n\mathbb{C}$. In fact, each of these subgroups is connected, and these inclusions induce isomorphisms of their fundamental groups.

Exercise 23.6. Prove these assertions by finding compatible actions of the subgroups on appropriate manifolds. Alternatively, observe that in each case the compact subgroup in question is just the subgroup of G preserving a Hermitian form on $\mathbb{C}^n$ or $\mathbb{C}^{2n}$, and use Gram–Schmidt to give a retraction of G onto the subgroup.

Now, by Proposition 23.1 the simply-connected complex Lie groups corresponding to the Lie algebras $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_m\mathbb{C}$ are

$$\tilde{G} = \mathrm{SL}_n\mathbb{C}, \quad \mathrm{Sp}_{2n}\mathbb{C}, \quad \text{and } \mathrm{Spin}_m\mathbb{C}.$$

We also know the center $Z(\tilde{G})$ of each of these groups. From Lecture 7 we also know the other connected groups with these Lie algebras:

- The complex Lie groups with Lie algebra $\mathfrak{sl}_n\mathbb{C}$ are $\mathrm{SL}_n\mathbb{C}$ and quotients of $\mathrm{SL}_n\mathbb{C}$ by subgroups of the form $\{e^{2\pi l i/m} \cdot I\}_l$ for m dividing n (in particular, if n is prime the only such groups are $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{PSL}_n\mathbb{C}$).
- The complex Lie groups with Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ are $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathrm{PSp}_{2n}\mathbb{C}$.
- The complex Lie groups with Lie algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ are $\mathrm{Spin}_{2n+1}\mathbb{C}$ and $\mathrm{SO}_{2n+1}\mathbb{C}$.
  and
- The complex Lie groups with Lie algebra $\mathfrak{so}_{2n}\mathbb{C}$ are $\mathrm{Spin}_{2n}\mathbb{C}$, $\mathrm{SO}_{2n}\mathbb{C}$ and $\mathrm{PSO}_{2n}\mathbb{C}$; in addition, if n is even, there are two other groups covered doubly by $\mathrm{Spin}_{2n}\mathbb{C}$ and covering doubly $\mathrm{PSO}_{2n}\mathbb{C}$ [cf. Exercise 20.36].

These are called the *classical groups*. In the cases where we have observed coincidences of Lie algebras, we have the following isomorphisms of groups:

$$\mathrm{Spin}_3\mathbb{C} \cong \mathrm{SL}_2\mathbb{C} \quad \text{and} \quad \mathrm{SO}_3\mathbb{C} \cong \mathrm{PSL}_2\mathbb{C};$$

$$\mathrm{Spin}_4\mathbb{C} \cong \mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C} \quad \text{and} \quad \mathrm{PSO}_4\mathbb{C} \cong \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C};$$

$$\mathrm{Spin}_5\mathbb{C} \cong \mathrm{Sp}_4\mathbb{C} \quad \text{and} \quad \mathrm{SO}_5\mathbb{C} \cong \mathrm{PSp}_4\mathbb{C};$$

and

$$\mathrm{Spin}_6\mathbb{C} \cong \mathrm{SL}_4\mathbb{C} \quad \text{and} \quad \mathrm{PSO}_6\mathbb{C} \cong \mathrm{PSL}_4\mathbb{C}.$$

Note that in the first case $n = 4$ where there is an intermediate subgroup between $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{PSL}_n\mathbb{C}$, the subgroup in question is interesting: it turns out to be $\mathrm{SO}_6\mathbb{C}$. In general, however, these intermediate groups seldom arise.

Consider now representations of these classical groups. According to the basic result of Lecture 7, representations of a complex Lie algebra $\mathfrak{g}$ will correspond exactly to representations of the associated simply connected Lie group $\tilde{G}$: specifically, for any representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$$

of $\mathfrak{g}$, setting

$$\tilde{\rho}(\exp(X)) = \exp(\rho(X))$$

determines a well-defined homomorphism

$$\tilde{\rho}\colon \tilde{G} \to \mathrm{GL}(V).$$

For any other group with algebra $\mathfrak{g}$, given as the quotient $\tilde{G}/C$ of $\tilde{G}$ by a subgroup $C \subset Z(\tilde{G})$, the representations of G are simply the representations of $\tilde{G}$ trivial on C. It is therefore enough to see which of the representations of the classical Lie algebras described in Part III are trivial on which subgroups $C \subset Z(\tilde{G})$.

This turns out to be very straightforward. To begin with, we observe that the center of each group G with Lie algebra $\mathfrak{g}$ lies in the image of the chosen Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ under the exponential map. It will therefore be enough to know when $\exp(\rho(X)) = I$ for $X \in \mathfrak{h}$; and since the representations ρ of $\mathfrak{g}$ are particularly simple on $\mathfrak{h}$ this presents no difficulty.

What we do have to do first is to describe the restriction of the exponential map to $\mathfrak{h}$, so that we can say which elements of $\mathfrak{h}$ exponentiate to elements of $Z(\tilde{G})$. For the groups that are given as matrix groups, this will all be perfectly obvious, but for the spin groups we will need to do a little calculation. We will also want to describe the *Cartan subgroup* H of each of the classical groups G, which is the connected subgroup whose Lie algebra is the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For $G = \mathrm{SL}_n\mathbb{C}$, H is just the diagonal matrices in G, i.e.,

$$H = \{\mathrm{diag}(z_1, \ldots, z_n)\colon z_1 \cdot \ldots \cdot z_n = 1\}.$$

Similarly in $\mathrm{Sp}_{2n}\mathbb{C}$ or $\mathrm{SO}_{2n}\mathbb{C}$, $H = \{\mathrm{diag}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})\}$, whereas in $\mathrm{SO}_{2n+1}\mathbb{C}$, $H = \{\mathrm{diag}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1)\}$. In each of these cases the exponential mapping from $\mathfrak{h}$ to H is just the usual exponentiation of diagonal matrices.

To calculate the exponential mapping for $\mathrm{Spin}_m\mathbb{C}$, we need to describe the elements in $\mathrm{Spin}_m\mathbb{C}$ that lie over the diagonal matrices in $\mathrm{SO}_m\mathbb{C}$. This is not a difficult task. Calculating as in §20.2, we find that for any nonzero complex number z and any $1 \le j \le n$, and with $m = 2n + 1$ or $m = 2n$, the elements

$$w_j(z) = \frac{1}{2}(ze_j \cdot e_{n+j} + z^{-1}e_{n+j} \cdot e_j) = z^{-1} + \left(\frac{z - z^{-1}}{2}\right)e_j \cdot e_{n+j} \qquad (23.7)$$

in the Clifford algebra are in fact elements of $\mathrm{Spin}_m\mathbb{C}$. Moreover, if $\rho\colon \mathrm{Spin}_m\mathbb{C} \to \mathrm{SO}_m\mathbb{C}$ is the covering, the image $\rho(w_j(z))$ is the diagonal matrix whose jth entry is z^2, $(n + j)$th entry is z^{-2}, and other diagonal entries are 1. These elements $w_j(z)$ also commute with each other, so for any nonzero complex numbers $z_1, \ldots, z_n$ we can define

$$w(z_1, \ldots, z_n) = w_1(z_1) \cdot w_2(z_2) \cdot \ldots \cdot w_n(z_n). \qquad (23.8)$$

Then $\rho(w(z_1, \ldots, z_n)) = \mathrm{diag}(z_1^2, \ldots, z_n^2, z_1^{-2}, \ldots, z_n^{-2})$ if $m = 2n$, while if $m = 2n + 1$, we get the same diagonal matrix but with a 1 at the end.

Let $H_i = E_{i,i} - E_{n+i,n+i}$, the usual basis for $\mathfrak{h} \subset \mathfrak{so}_m\mathbb{C}$.

Lemma 23.9. *For any complex numbers* $a_1, \ldots, a_n$,

$$\exp(a_1H_1 + \cdots + a_nH_n) = w(e^{a_1/2}, \ldots, e^{a_n/2})$$

in $\mathrm{Spin}_m\mathbb{C}$.

PROOF. Since the map $\exp\colon \mathfrak{h} \to \mathrm{Spin}_m\mathbb{C}$ is determined by the facts that it is continuous, it takes 0 to 1, and its composite with ρ is the exponential for $\mathrm{SO}_m\mathbb{C}$, this follows from the preceding formulas. □

Exercise 23.10*. Show that $\exp(\sum a_jH_j) = 1$ if and only if each a_j is in $2\pi i\mathbb{Z}$ and $\sum a_j \in 4\pi i\mathbb{Z}$.

We see also that $\exp(\mathfrak{h})$ contains the center of $\mathrm{Spin}_m\mathbb{C}$. Indeed, $-1 = w(-1, 1, \ldots, 1)$, and if m is even, the other central elements are $\pm\omega$, with $\omega = w(i, \ldots, i)$, as we calculated in Exercise 20.36. (This, of course, also contains the fact that there is a path between 1 and -1, proving again that $\mathrm{Spin}_m\mathbb{C}$ is connected.)

Exercise 23.11*. Verify for all the classical groups G that: (i) $H = \exp(\mathfrak{h})$ is a closed subgroup of G that contains the center of G; (ii) the map of fundamental groups $\pi_1(H, e) \to \pi_1(G, e)$ is surjective; (iii) for any connected covering $\pi\colon G' \to G$, $\pi^{-1}(H)$ is connected and is the Cartan subgroup of G'.

Now let $G = \tilde{G}/C$ be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$. Choose an ordering of the roots, and let Γ_λ be the irreducible representation of $\mathfrak{g}$ with highest weight λ. The basic fact that we need is

Lemma 23.12. *The representation Γ_λ is a representation of $G = \tilde{G}/C$ if and only if*

$$\lambda(X) \in 2\pi i\mathbb{Z} \quad \textit{whenever } \exp(X) \in C.$$

Proof. The representation Γ_λ is a representation of G when $g \cdot v = v$ for all $g \in C$, where v is a highest weight vector in Γ_λ. Since $\exp(\mathfrak{h})$ contains C, this says $\exp(X) \cdot v = v$ for all $X \in \mathfrak{h}$ such that $\exp(X) \in C$. Now by the naturality of the exponential map, and since $X \cdot v = \lambda(X)v$ for $X \in \mathfrak{h}$, we have $\exp(X) \cdot v = e^{\lambda(X)}v$. Hence the condition is that $e^{\lambda(X)}v = v$, or that $e^{\lambda(X)} = 1$ if $\exp(X) \in C$, which is the displayed criterion. □

Let us work this out explicitly for each of the classical groups. It may help to introduce a notation for the irreducible representations which, among other virtues, allows some common terminology in the various cases. Note that for each of $\mathfrak{sl}_{n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, and $\mathfrak{so}_{2n+1}$ the root space $\mathfrak{h}^*$ is spanned by weights we have called $L_1, \ldots, L_n$, so a weight can be written uniquely in form $\lambda_1 L_1 + \cdots + \lambda_n L_n$. We may sometimes write λ in place of the weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$. In the rest of this lecture at least, we write Γ_λ for the irreducible representation with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$. Note that by our choice of Weyl chambers the highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ that arise satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \quad \text{for } \mathfrak{sl}_{n+1}, \mathfrak{sp}_{2n}, \text{and } \mathfrak{so}_{2n+1},$$

where the λ_i are all integers in the first two cases, and for $\mathfrak{so}_{2n+1}$ they are either all integers or all half-integers; and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \geq 0 \quad \text{for } \mathfrak{so}_{2n},$$

with the λ_i all integers or all half-integers.

Proposition 23.13. *For each subgroup C of the center of $\tilde{G}$, the representation Γ_λ is a representation of $\tilde{G}/C$ precisely under the following conditions:*

(i) $\tilde{G} = \mathrm{SL}_{n+1}\mathbb{C}$, *$C$ has order m dividing $n+1$: $\sum \lambda_j \equiv 0 \bmod(m)$.*
(ii) $\tilde{G} = \mathrm{Sp}_{2n}\mathbb{C}$, *$C = \{\pm 1\}$: $\sum \lambda_j$ is even.*
(iii) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$ *or* $\mathrm{Spin}_{2n+1}\mathbb{C}$, *$C = \{\pm 1\}$: all λ_i are integers.*
(iv) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$, *$C = \{\pm 1, \pm\omega\}$: all λ_i are integers, $\sum \lambda_j$ is even.*
(v) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$, *$n$ even, $C = \{1, \omega\}$: $\sum \lambda_j$ is an even integer; and for $C = \{1, -\omega\}$: $\sum \lambda_j - n/2$ is an odd integer.*

In particular, representations of $\mathrm{PSL}_{n+1}\mathbb{C}$ are given by partitions λ with $\sum \lambda_j \equiv 0 \bmod(n+1)$, and those for $\mathrm{PSp}_{2n}\mathbb{C}$ have $\sum \lambda_j$ even. Case (iii) verifies what we saw in Lecture 19 about representations of $\mathrm{SO}_m\mathbb{C}$. Representations of $\mathrm{PSO}_m\mathbb{C}$ correspond to integral partitions λ with $\sum \lambda_j$ even.

Proof. With the preceding lemma and the explicit description of everything in sight, the calculations are routine. In case (i), for example, a generator for

C is of the form $\exp(X)$, with

$$X = (2\pi i/m)\left(\sum_{j=1}^{n} E_{j,j} - nE_{n+1,n+1}\right),$$

and so $\lambda(X) = (2\pi i/m)(\sum \lambda_j)$ will be a multiple of $2\pi i$ exactly when $\sum \lambda_j$ is divisible by m. For $\mathrm{Sp}_{2n}\mathbb{C}$, $\exp(X) = -1$ when $X = \pi i(\sum H_j)$, so $\lambda(X) = \pi i \sum \lambda_j$, and (ii) follows. The calculations are similar for $\mathrm{Spin}_m\mathbb{C}$, noting that $\exp(2\pi i(H_1) = -1$ and $\exp(\pi i(\sum H_j)) = \omega$. □

By way of an example, recall that any irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is of the form $\mathrm{Sym}^k V$, where V is the standard two-dimensional representation. Any such representation, of course, lifts to the group $\mathrm{SL}_2\mathbb{C}$; but it lifts to $\mathrm{PSL}_2\mathbb{C} \cong \mathrm{SO}_3\mathbb{C}$ if and only if k is even (in particular, the "standard" representation of $\mathrm{SO}_3\mathbb{C}$ on $\mathbb{C}^3$ is the symmetric square $\mathrm{Sym}^2 V$). For another example, we have seen that any irreducible representation of $\mathfrak{sp}_4\mathbb{C}$ may be found in a tensor product $\mathrm{Sym}^k V \otimes \mathrm{Sym}^l W$, where V is the standard four-dimensional representation of $\mathfrak{sp}_4\mathbb{C}$ and $W \subset \bigwedge^2 V$ the complement of the trivial one-dimensional representation. All such representations lift to $\mathrm{Sp}_4\mathbb{C}$, but they lift to $\mathrm{PSp}_4\mathbb{C} \cong \mathrm{SO}_5\mathbb{C}$ if and only if k is even—equivalently, if they are contained in a representation of the form $\mathrm{Sym}^l W \otimes \mathrm{Sym}^k(\bigwedge^2 W)$, where W is the "standard" representation of $\mathrm{SO}_5\mathbb{C}$.

Exercise 23.14. Show that each of these semisimple complex Lie groups G has a finite-dimensional faithful representation.

The result of the proposition can be put in a more formal setting, which brings out a feature that our alert reader has surely noticed: the center of the simply-connected form of $\mathfrak{g}$ is isomorphic to the quotient group Λ_W/Λ_R of the weight lattice modulo the root lattice. We note first that this abelian group Λ_W/Λ_R is *finite*. We have seen this for the classical Lie algebras. In general, we have

Lemma 23.15. *The group Λ_W/Λ_R is finite, of order equal to the determinant of the Cartan matrix.*

Proof. The simple roots α form a basis for the root lattice Λ_R. The corresponding elements H_α form a basis for

$$\Gamma_R = \mathbb{Z}\{H_\gamma : \gamma \in R\},$$

a lattice in $\mathfrak{h}$; this is proved in Appendix D.4. Since Λ_W is defined to be the lattice of elements of $\mathfrak{h}^*$ that take integral values on Γ_R, the determinant

$$\det(\alpha(H_\beta)) = \det(n_{\alpha\beta})$$

is the index $[\Lambda_W : \Lambda_R]$. □

In particular, for the exceptional groups, Λ_W/Λ_R is trivial for (G_2), (F_4), and (E_8), and cyclic of order two for (E_7) and order three for (E_6).

In fact, the center of the simply-connected group is naturally isomorphic to the *dual* of Λ_W/Λ_R. To express this, consider the natural dual of this last group. The lattice Γ_R defined in the preceding proof is a sublattice of the lattice

$$\Gamma_W = \{X \in \mathfrak{h}: \alpha(X) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Note that Λ_W was defined to be the lattice of elements of $\mathfrak{h}^*$ that take integral values on Γ_R. It follows formally from the definitions and the fact that Λ_W/Λ_R is finite that we have a perfect pairing

$$\Gamma_W/\Gamma_R \times \Lambda_W/\Lambda_R \to \mathbb{Q}/\mathbb{Z}, \qquad (X, \alpha) \mapsto \alpha(X).$$

The claim is that there is a natural isomorphism from Γ_W/Γ_R to the center of $\tilde{G}$, which is given by the exponential. More precisely, let $e_G: \mathfrak{h} \to H \subset G$ be the homomorphism defined by

$$e_G(X) = \exp(2\pi i X).$$

We claim that when $G = \tilde{G}$ is the simply-connected group, $\operatorname{Ker}(e_{\tilde{G}}) = \Gamma_R$ and $e_{\tilde{G}}(\Gamma_W)$ is the center of $\tilde{G}$, from which it follows that $e_{\tilde{G}}$ induces an isomorphism

$$\Gamma_W/\Gamma_R \cong Z(\tilde{G}).$$

More generally, for any $G = \tilde{G}/C$, define a lattice $\Gamma(G)$ between Γ_R and Γ_W by

$$\Gamma(G) = \operatorname{Ker}(e_G).$$

Then e_G determines an isomorphism

$$\Gamma_W/\Gamma(G) \cong Z(G).$$

We may thus state our result as

Theorem 23.16. *There is a one-to-one correspondence between connected Lie groups G with the Lie algebra $\mathfrak{g}$ and lattices $\Lambda \subset \mathfrak{h}^*$ such that*

$$\Lambda_R \subset \Lambda \subset \Lambda_W.$$

The correspondence is given by associating to a group G the lattice dual to the kernel of the exponential map exp: $\mathfrak{g} \to G$; *in particular, the largest lattice Λ_W corresponds to the simply-connected group, the smallest Λ_R to the adjoint group with no center. In terms of this correspondence, the irreducible representation V_λ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ will lift to a representation of the group G corresponding to $\Lambda \subset \mathfrak{h}^*$ if and only if $\lambda \in \Lambda$.*

Note also that

$$H = \mathfrak{h}/\Gamma(G) \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*,$$

with $n = \dim_{\mathbb{C}}\mathfrak{h}$ copies of $\mathbb{C}^*$.

Exercise 23.17*. Show that these claims follow formally from what we have seen: that the image of the exponential map contains the center, and that for any weight α there is a representation V of $\mathfrak{g}$ whose weight space V_α is not zero. Show also that e_G determines an isomorphism $\Gamma(G)/\Gamma_R \cong \pi_1(G)$. In diagram form,

$$\begin{array}{ccc} \Gamma_W & & G_0 \\ \cup & \text{Center}(G) & \uparrow \\ \Gamma(G) & & G \\ \cup & \pi_1(G) & \uparrow \\ \Gamma_R & & \tilde{G} \end{array}$$

Exercise 23.18. Find the kernels of each of the spin and half-spin representations $\mathrm{Spin}_m\mathbb{C} \to \mathrm{GL}(S)$ and $\mathrm{Spin}_m\mathbb{C} \to \mathrm{GL}(S^{\pm})$.

Exercise 23.19*. Classify the irreducible representations of the full orthogonal group $\mathrm{O}_m\mathbb{C}$.

Note that by our analysis of the Lie algebra $\mathfrak{g}_2$ there is a unique group G_2 with this Lie algebra, which is simultaneously the simply-connected and adjoint forms; the representations of this group are exactly those of the algebra $\mathfrak{g}_2$. The same is true for the Lie algebras of type (F_4) and (E_8), while (E_7) and (E_6) each have two associated groups, an adjoint one with fundamental group $\mathbb{Z}/2$ and $\mathbb{Z}/3$, and a simply-connected form with center $\mathbb{Z}/2$ and $\mathbb{Z}/3$ respectively.

It may be worth pointing out that each complex simple Lie group G can be realized as a closed subgroup defined by polynomial equations in some general linear group, i.e., that G is an *affine algebraic group.* Every irreducible representation $G \to \mathrm{GL}(V)$ is also defined by polynomials in appropriate coordinates. This explains why the whole subject can be developed from the point of view of algebraic groups, as in [Bor1] and [Hu2].

The Weyl group $\mathfrak{W}$, which we defined as a subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$, can be interpreted in terms of any connected Lie group G with Lie algebra $\mathfrak{g}$. Let H be the Cartan subgroup corresponding to $\mathfrak{h}$, and let $N(H)$ be the normalizer:

$$N(H) = \{g \in G: gHg^{-1} = H\}.$$

We have homomorphisms:

$$N(H) \to \mathrm{Aut}(H) \to \mathrm{Aut}(\mathfrak{h}) \to \mathrm{Aut}(\mathfrak{h}^*),$$

the first defined by conjugation, the second by differentiation at the identity, and the third using the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Killing form. Fact 14.11 can be sharpened to the claim that this map determines an *isomorphism*

$$N(H)/N \xrightarrow{\cong} \mathfrak{W}. \tag{23.20}$$

When G is the adjoint form of the Lie algebra, this isomorphism is proved in Appendix D. The general case follows, using:

Exercise 23.21. Show that if $\pi: G' \to G$ is a connected covering, with Cartan subgroups $H' = \pi^{-1}(H)$, then the induced map $N(H')/H' \to N(H)/H$ is an isomorphism.

Exercise 23.22. For each of the classical groups, and each simple root α, find an element in $N(H)$ that maps to the reflection W_α in $\mathfrak{W}$.

§23.2. Representation Rings and Characters

Just as with finite groups, we can form the representation ring R of a semisimple Lie algebra or Lie group: take the free abelian group on the isomorphism classes $[V]$ of finite-dimensional representations V, and divide by the relations $[V] = [V'] + [V'']$ whenever $V \cong V' \oplus V''$. By the complete reducibility of representations, it follows as before that R is a free abelian group on the classes $[V]$ of irreducible representations. Again, the tensor product of representations makes R into a ring: $[V]\cdot[W] = [V \otimes W]$. Many of our questions about decomposing representations and tensor products of representations can be nicely encoded by describing R more fully. We do this first for the Lie algebras.

For a semisimple Lie algebra $\mathfrak{g}$, let $\Lambda = \Lambda_W$ be the weight lattice, and let $\mathbb{Z}[\Lambda]$ be the integral group ring on the abelian group Λ. We write $e(\lambda)$ for the basis element of $\mathbb{Z}[\Lambda]$ corresponding to the weight λ; for now at least these are just formal symbols, having nothing to do with exponentials (but see (23.40)). Elements of $\mathbb{Z}[\Lambda]$ are expressions of the form $\sum n_\lambda e(\lambda)$, i.e., they assign an integer n_λ to each weight λ, with all but a finite number being zero. So $\mathbb{Z}[\Lambda]$ is a natural carrier for the information about multiplicities of representations. Define a *character homomorphism*

$$\text{Char}: R(\mathfrak{g}) \to \mathbb{Z}[\Lambda] \tag{23.23}$$

by the formula $\text{Char}[V] = \sum \dim(V_\lambda)e(\lambda)$, where V_λ is the weight space of V for the weight λ and $\dim(V_\lambda)$ its multiplicity. This is clearly an additive homomorphism.

The first assertion about this character map is that it is *injective*. This comes down to the fact that a representation is determined by the multiplicities of its weight spaces, which is something we saw in Lecture 14.

The product in the group ring $\mathbb{Z}[\Lambda]$ is determined by $e(\alpha)\cdot e(\beta) = e(\alpha + \beta)$. We claim next that Char is a *ring homomorphism*. This comes from the familiar fact that

$$(V \otimes W)_\lambda = \bigoplus_{\mu+\nu=\lambda} V_\mu \otimes W_\nu.$$

The Weyl group $\mathfrak{W}$ acts on $\mathbb{Z}[\Lambda]$, and a third simple claim is that the image of Char is contained in the ring of invariants $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$. This comes down to the fact that, for an irreducible (and hence for any) representation V, the weight spaces obtained by reflecting in walls of the Weyl chambers all have the same dimension.

Let $\omega_1, \ldots, \omega_n$ be a set of fundamental weights; as we have seen, these are the first weights along edges of a Weyl chamber, and they are free generators for the lattice Λ. Let $\Gamma_1, \ldots, \Gamma_n$ be the classes in $R(\mathfrak{g})$ of the irreducible representations with highest weights $\omega_1, \ldots, \omega_n$.

Theorem 23.24. (a) *The representation ring* $R(\mathfrak{g})$ *is a polynomial ring on the variables* $\Gamma_1, \ldots, \Gamma_n$.

(b) *The homomorphism* $R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}$ *is an isomorphism.*

In particular, this says that $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ is a polynomial ring on the variables $\mathrm{Char}(\Gamma_1), \ldots, \mathrm{Char}(\Gamma_n)$. In fact, the theorem is equivalent to this assertion, since if we take variables $U_1, \ldots, U_n$ and map the polynomial ring on the U_i to $R(\mathfrak{g})$ by sending U_i to Γ_i, we have

$$\mathbb{Z}[U_1, \ldots, U_n] \to R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}.$$

If the composite is an isomorphism, the second being injective, both must be isomorphisms, which is what the theorem says.

In spite of its fancy appearance, we will see that the theorem follows quite easily from what we know about the action of the Weyl group $\mathfrak{W}$ on the weights.

For any $P \in \mathbb{Z}[\Lambda]$ let us say that α is a *highest weight* for P if the coefficient of $e(\alpha)$ in P is nonzero, and, with a chosen ordering of weights as before, α is the largest such weight. We first observe that if P is invariant under $\mathfrak{W}$, then the highest weight for P is in $\mathscr{W} \cap \Lambda$, where $\mathscr{W}$ is our chosen (closed) Weyl chamber. In general, weights in $\mathscr{W} \cap \Lambda$ are often referred to as *dominant weights.*

Now suppose $\{P_\lambda\}$ is any collection of elements in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$, one for each dominant weight λ, such that P_λ has highest weight λ and the coefficient of $e(\lambda)$ is 1. We claim that the P_λ form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ over $\mathbb{Z}$. This is easy to see and is the same argument used in the theory of symmetric polynomials in any algebra text: given P with highest weight λ, if the coefficient of $e(\lambda)$ is m, then $P - mP_\lambda$ is invariant whose highest weight is lower, and one continues inductively until one reaches weight zero, i.e., the constants.

Let $P_i = \mathrm{Char}(\Gamma_i)$, which has highest weight ω_i, and suppose the coefficient of $e(\omega_i)$ is 1. Since any weight $\lambda \in \mathscr{W} \cap \Lambda$ can be uniquely expressed in the form $\lambda = \sum m_i\omega_i$, for some non-negative integers m_i, and the highest weight of $\prod (P_i)^{m_i}$ is $\sum m_i\omega_i$, it follows that the monomials $\prod (P_i)^{m_i}$ in $P_1, \ldots, P_n$ form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$. This says precisely that $\mathbb{Z}[P_1, \ldots, P_n] = \mathbb{Z}[\Lambda]^{\mathfrak{W}}$, and completes the proof. □

Let us work this out concretely for each of our cases $\mathfrak{sl}_{n+1}\mathbb{C}$, $\mathfrak{sp}_n\mathbb{C}$, $\mathfrak{so}_{2n+1}\mathbb{C}$, and $\mathfrak{so}_{2n}\mathbb{C}$. Each lattice Λ contains weights we have called $L_1, \ldots, L_n$; in the first case we also have L_{n+1} with $L_1 + \cdots + L_{n+1} = 0$. We set

$$x_i = e(L_i), \qquad x_i^{-1} = e(-L_i) \in \mathbb{Z}[\Lambda]. \tag{23.25}$$

Note that in case $L_1, \ldots, L_n$ is a basis for Λ, then

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}] = \mathbb{Z}[x_1, \ldots, x_n, (x_1 \cdot \ldots \cdot x_n)^{-1}]$$

as a subring of the field $\mathbb{Q}(x_1, \ldots, x_n)$.

(A_n) For $\mathfrak{sl}_{n+1}\mathbb{C}$, fundamental weights are

$$L_1, \quad L_1 + L_2, \quad L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_n,$$

corresponding to the irreducible representations V, $\wedge^2 V$, ..., $\wedge^n V$, with $V = \mathbb{C}^{n+1}$ the standard representation. The character of $\wedge^k V$ is $\sum e(\alpha)$, the sum over all α that are sums of k different L_i for $1 \le i \le n+1$. So $\operatorname{Char}(\wedge^k V) = A_k$, where A_k is the kth elementary symmetric function of $x_1, \ldots, x_{n+1}$. The Weyl group is the symmetric group $\mathfrak{S}_{n+1}$, acting by permutation on the indices, so the theorem in this case says that

$$R(\mathfrak{sl}_{n+1}) = \mathbb{Z}[\Lambda]^{\mathfrak{S}_{n+1}} = \mathbb{Z}[A_1, \ldots, A_n]. \tag{23.26}$$

Note that $\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, \ldots, x_n, x_{n+1}]/(x_1 \cdot \ldots \cdot x_{n+1} - 1)$, so $\mathbb{Z}[\Lambda]$ has an additive basis consisting of all monomials x^α, with α an n-tuple of non-negative integers, but with not all α_i positive.

(C_n) For $\mathfrak{sp}_{2n}\mathbb{C}$, the lattice Λ and fundamental weights have the same description as in the preceding case. The corresponding irreducible representations are the kernels $V^{(k)}$ of the contraction maps $\wedge^k V \to \wedge^{k-2} V$, with now $V = \mathbb{C}^{2n}$ the standard representation, $k = 1, \ldots, n$. The character of $\wedge^k V$ is $\sum e(\alpha)$, the sum over all α that are sums of k different $\pm L_i$ for $1 \le i \le n$. The character $\operatorname{Char}(\wedge^k V)$ is thus the elementary symmetric polynomial C_k in the variables $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$. The theorem then says that

$$\begin{aligned} R(\mathfrak{sp}_{2n}\mathbb{C}) &= \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[C_1, C_2 - 1, C_3 - C_1, \ldots, C_n - C_{n-2}] \\ &= \mathbb{Z}[C_1, C_2, C_3, \ldots, C_n]. \end{aligned} \tag{23.27}$$

(B_n) For $\mathfrak{so}_{2n+1}\mathbb{C}$, Λ is spanned by the L_i together with $\frac{1}{2}(L_1 + \cdots + L_n)$. The fundamental representations are V, $\wedge^2 V$, ..., $\wedge^{n-1} V$, and the spin representation S. The character of $\wedge^k V$ is the kth elementary symmetric function of the $2n+1$ elements $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$, and 1; denote this by B_k. The character of S, which we denote by B, is the sum $\sum x_1^{\pm 1/2} \cdot \ldots \cdot x_n^{\pm 1/2}$, where

$$x_i^{+1/2} = e(L_i/2), \qquad x_i^{-1/2} = e(-L_i/2). \tag{23.28}$$

So B is the nth elementary symmetric polynomial in the variables $x_i^{+1/2} + x_i^{-1/2}$. Therefore,

$$R(\mathfrak{so}_{2n+1}\mathbb{C}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[B_1, \ldots, B_{n-1}, B]. \tag{23.29}$$

(D_n) For $\mathfrak{so}_{2n}\mathbb{C}$, Λ and $\mathbb{Z}[\Lambda]$ are the same as in the preceding case. The fundamental representations are V, $\wedge^2 V$, ..., $\wedge^{n-2} V$, and the half-spin representations S^+ and S^-. The character of $\wedge^k V$, denoted D_k, is the kth elementary symmetric function of the $2n$ elements $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$. The character $D^{\pm}$ of $S^{\pm}$ is the sum $\sum x_1^{\pm 1/2} \cdot \ldots \cdot x_n^{\pm 1/2}$, where the number of plus signs is even or odd according to the sign. We have

$$R(\mathfrak{so}_{2n}\mathbb{C}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[D_1, \ldots, D_{n-2}, D^+, D^-]. \tag{23.30}$$

Exercise 23.31*. (a) Prove the following relation in $R(\mathfrak{so}_{2n+1}\mathbb{C})$:

$$B^2 = B_n + \cdots + B_1 + 1,$$

corresponding to the isomorphism

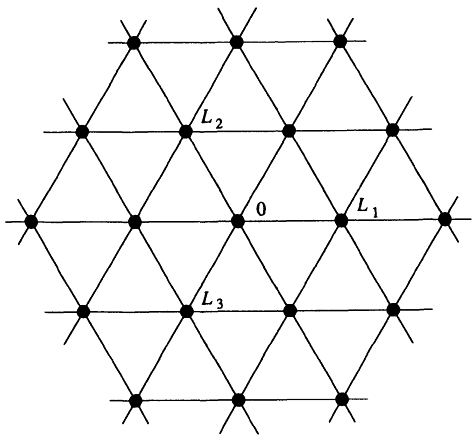
$$S \otimes S \cong \wedge^n V \oplus \cdots \oplus \wedge^1 V \oplus \wedge^0 V.$$

This describes $R(\mathfrak{so}_{2n+1}\mathbb{C})$ as a quadratic extension of the ring $\mathbb{Z}[B_1, \ldots, B_n]$.

(b) Let D_n^+ (respectively, D_n^-) be the character of the representation whose highest weight is twice that of D^+ (resp., D^-), so that, for example, the sum of the representations D_n^+ and D_n^- is $\wedge^n V$. Prove the relations in $R(\mathfrak{so}_{2n}\mathbb{C})$:

$$\begin{aligned}
D^+ \cdot D^+ &= D_n^+ + D_{n-2} + D_{n-4} + \cdots,\\
D^- \cdot D^- &= D_n^- + D_{n-2} + D_{n-4} + \cdots,\\
D^+ \cdot D^- &= D_{n-1} + D_{n-3} + D_{n-5} + \cdots.
\end{aligned}$$

We can likewise describe the representation ring for $\mathfrak{g}_2$. Here, we may take as generators for the weight lattice the weights L_1 and L_2 as pictured in the diagram



and correspondingly write $\mathbb{Z}[\Lambda]$ as $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}]$, where $x_i = e(L_i)$. It will be a little more symmetric to introduce $L_3 = -L_1 - L_2$ as pictured and $x_3 = x_1^{-1} \cdot x_2^{-1} = e(L_3)$, and write

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, x_2, x_3]/(x_1 x_2 x_3 - 1).$$

In these terms the Weyl group is the group $\mathfrak{W}$ generated by the symmetric group $\mathfrak{S}_3$ permuting the variables x_i and the involution sending each x_i to x_i^{-1}. The standard representation has weights $\pm L_i$ and 0, and so has character

$$A = A(x_1, x_2, x_3) = 1 + x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_3 + x_3^{-1}.$$

Similarly, the adjoint representation has weights $\pm L_i$, $\pm(L_i - L_j)$, and 0 (taken twice); its character is

$$B = A(x_1, x_2, x_3) + A(x_1/x_2, x_2/x_3, x_3/x_1).$$

The theorem thus implies in this case the equality

$$R(\mathfrak{g}_2) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[A, B]. \tag{23.32}$$

Exercise 23.33. Verify directly the statement that any element of $\mathbb{Z}[x_1, x_2, x_3]/(x_1x_2x_3 - 1)$ invariant under the group $\mathfrak{W}$ as described is in fact a polynomial in A and B.

Similarly we can define the representation ring $R(G)$ of a semisimple group G. When G is the simply-connected form of its Lie algebra $\mathfrak{g}$, $R(G) = R(\mathfrak{g})$, so $R(\mathrm{SL}_n\mathbb{C})$, $R(\mathrm{Sp}_{2n}\mathbb{C})$, $R(\mathrm{Spin}_{2n+1}\mathbb{C})$, and $R(\mathrm{Spin}_{2n}\mathbb{C})$ are given by (23.26), (23.27), (23.29), and (23.30). In general, $R(G)$ is a subring of $R(\mathfrak{g})$; we can read off which subring by looking at Proposition 23.13. We have, in fact,

$$R(\mathrm{SO}_{2n+1}\mathbb{C}) = \mathbb{Z}[B_1, \dots, B_n]; \tag{23.34}$$

$$R(\mathrm{SO}_{2n}\mathbb{C}) = \mathbb{Z}[D_1, \dots, D_{n-1}, D_n^+, D_n^-], \tag{23.35}$$

with D_n^+ and D_n^- as in Exercise 23.31. But this time there is one relation:

$$\begin{aligned}(D_n^+ + D_{n-2} + D_{n-4} + \cdots + 1)(D_n^- + D_{n-2} + D_{n-4} + \cdots + 1)\\ = (D_{n-1} + D_{n-3} + \cdots +)^2.\end{aligned}$$

Exercise 23.36*.

(a) Prove (23.34).
(b) Show that the relation in (23.35) comes from Exercise 23.31(b). Show that $R(\mathrm{SO}_{2n}\mathbb{C})$ is the polynomial ring in the $n + 1$ generators shown, modulo the ideal generated by the one polynomial indicated.
(c) Describe the representation rings for the other groups with these simple Lie algebras.
(d) Prove the isomorphism

$$R(\mathrm{GL}_n\mathbb{C}) = \mathbb{Z}[E_1, \dots, E_n, E_n^{-1}],$$

where the E_k are the elementary symmetric functions of $x_1, \dots, x_n$.

Exercise 23.37*. (a) Show that the image of $R(\mathrm{O}_m\mathbb{C})$ in $R(\mathrm{SO}_m\mathbb{C})$ is the polynomial ring $\mathbb{Z}[B_1, \dots, B_n]$ if $m = 2n + 1$, and $\mathbb{Z}[D_1, \dots, D_n]$ if $m = 2n$.

(b) Show that

$$R(\mathrm{O}_{2n+1}\mathbb{C}) = R(\mathrm{SO}_{2n+1}\mathbb{C}) \otimes R(\mathbb{Z}/2)$$
$$= \mathbb{Z}[B_1, \ldots, B_n, B_{2n+1}]/((B_{2n+1})^2 - 1)$$

and

$$R(\mathrm{O}_{2n}\mathbb{C}) = \mathbb{Z}[D_1, \ldots, D_n, D_{2n}]/I,$$

where I is the ideal generated by $(D_{2n})^2 - 1$ and $D_n D_{2n} - D_n$.

Exercise 23.38*. The mapping that takes a representation V to its dual V^* induces an involution of the representation ring: $[V]^* = [V^*]$. The ring $\mathbb{Z}[\Lambda]$ has an involution determined by $(e(\lambda))^* = e(-\lambda)$. Show that the character homomorphism commutes with these involutions. Show that for $\mathfrak{sl}_{n+1}$, $(A_k)^* = A_{n+1-k}$; for $\mathfrak{so}_{2n+1}\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_{2n}\mathbb{C}$ for n even, the involution is the identity; while for $\mathfrak{so}_{2n}\mathbb{C}$ with n odd, $(D_k)^* = D_k$, $(D^+)^* = D^-$, $(D^-)^* = D^+$. Deduce that all representations of all symplectic and orthogonal groups are self-dual. Note that when $*$ is the identity, all representations are self-dual. In the other cases, compute the duals of irreducible representations with given highest weight.

The following exercise deals with a special property of the representation rings of semisimple Lie groups and algebras.

Exercise 23.39*. The representation rings $R = R(\mathfrak{g})$ and $R(G)$ have another important structure: they are *λ-rings*. There are operators

$$\lambda^i\colon R(G) \to R(G), \quad i = 0, 1, 2, \ldots,$$

determined by $\lambda^i([V]) = [\wedge^i V]$ for any representation V.

(a) Show that this determines well-defined maps, satisfying $\lambda^0 = 1$, $\lambda^1 = \mathrm{Id}$, and

$$\lambda^i(x + y) = \sum_{i+j=k} \lambda^i(x)\cdot\lambda^j(y)$$

for any x and y in R. In fact, R is what is called a *special* λ-ring: there are formulas for $\lambda^i(x\cdot y)$ and $\lambda^i(\lambda^j(x))$, valid as if x and y could be written as sums of one-dimensional representations (see, e.g., [A-T]).

(b) Show that λ^i extends to $\mathbb{Z}[\Lambda]$, and use this to verify that $R(G)$ is a special λ-ring.

Define *Adams operators* $\psi^k\colon R \to R$ by $\psi^k(x) = P_k(\lambda^1 x, \ldots, \lambda^n x)$, where P_k is the expression for the kth power sum (cf. Exercise A.32) in terms of the elementary symmetric functions, $n \geq k$. Equivalently,

$$\psi^k(x) - \psi^{k-1}(x)\lambda^1(x) + \cdots + (-1)^k k\lambda^k(x) = 0.$$

(c) Show that, regarding R as the ring of functions on the group G, $(\psi^k x)(g) = x(g^k)$. Equivalently, $\psi^k(e(\lambda)) = e(k\lambda)$.

(d) Show that each ψ^k is a ring homomorphism, and $\psi^k \circ \psi^l = \psi^{k+l}$.
(e) Show that for a representation V,

$$\mathrm{Char}(\mathrm{Sym}^2 V) = \tfrac{1}{2}\,\mathrm{Char}(V)^2 + \tfrac{1}{2}\psi^2(\mathrm{Char}(V)),$$
$$\mathrm{Char}(\wedge^2 V) = \tfrac{1}{2}\,\mathrm{Char}(V)^2 - \tfrac{1}{2}\psi^2(\mathrm{Char}(V)).$$

Show that $\mathrm{Char}(\mathrm{Sym}^d V)$ and $\mathrm{Char}(\wedge^d V)$ can be written as polynomials in $\psi^k(\mathrm{Char}(V))$, $1 \le k \le d$.

Formal Characters and Actual Characters

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any representation V of $\mathfrak{g}$, the image of $[V] \in R(\mathfrak{g})$ in $\mathbb{Z}[\Lambda]$ is called the *formal character* of V. As it turns out, this formal character can be identified with the honest character of the corresponding representation of the group G, restricted to the Cartan subgroup H:

(23.40) *If* $\mathrm{Char}(V) = \sum m_\alpha e(\alpha)$ *is the formal character, and* $\exp(X)$ *is an element of* H*, then the trace of* $\exp(X)$ *on* V *is* $\sum m_\alpha e^{\alpha(X)}$.

This is simply because $\exp(X)$ acts on the weight space V_μ by multiplication by $e^{\mu(X)}$, as we have seen. In particular, a representation is determined by the character of its restriction to a Cartan subgroup.

Another common notation for this is to set $e(X) = \exp(2\pi i X)$, and $e(z) = \exp(2\pi i z)$. Then the trace of $e(X)$ is $\sum m_\alpha e(\alpha(X))$.

Exercise 23.41. As a function on H, the character of a representation is invariant under the Weyl group $\mathfrak{W} = N(H)/H$. Describe $R(G)$ as a ring of $\mathfrak{W}$-invariant functions on H.

This is also compatible with our descriptions of elements of $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ as Laurent polynomials in variables x_i or $x_i^{1/2}$. For $\mathrm{SL}_{n+1}\mathbb{C}$, for example, if the character $\mathrm{Char}(W)$ of a representation W is $P(x_1, \ldots, x_{n+1})$, the trace of the matrix $\mathrm{diag}(z_1, \ldots, z_{n+1})$ on V is $P(z_1, \ldots, z_{n+1})$. Similarly for the other groups, using the diagonal matrices described in the first section of this lecture. For the spin groups, the element $w(z_1, \ldots, z_n)$ defined in (23.8) has trace given by substituting z_i for $x_i^{1/2}$, and z_i^{-1} for $x_i^{-1/2}$ in the corresponding Laurent polynomial.

Exercise 23.42*. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two semisimple Lie algebras, show that

$$R(\mathfrak{g}_1 \times \mathfrak{g}_2) = R(\mathfrak{g}_1) \otimes R(\mathfrak{g}_2).$$

Exercise 23.43*. (a) For the natural inclusion $\mathfrak{sl}_n\mathbb{C} \subset \mathfrak{sl}_{n+1}\mathbb{C}$, restriction of representations gives a homomorphism $R(\mathfrak{sl}_{n+1}\mathbb{C}) \to R(\mathfrak{sl}_n\mathbb{C})$, which can be described by saying what happens to the polynomial generators. Since $\bigwedge^k(\mathbb{C}^n \oplus \mathbb{C}) = \bigwedge^k(\mathbb{C}^n) \oplus \bigwedge^{k-1}(\mathbb{C}^n)$, this is

$$A_k \mapsto A_k + A_{k-1}.$$

Give the analogous descriptions for the following inclusions:

$$\mathfrak{sp}_{2n-2}\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}, \qquad \mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}, \qquad \mathfrak{so}_{2n-1}\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C};$$

$$\mathfrak{sl}_n\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}, \qquad \mathfrak{sl}_n\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}, \qquad \mathfrak{sl}_n\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C};$$

$$\mathfrak{sp}_{2n}\mathbb{C} \subset \mathfrak{sl}_{2n}\mathbb{C}, \qquad \mathfrak{so}_{2n+1}\mathbb{C} \subset \mathfrak{sl}_{2n+1}\mathbb{C}, \qquad \mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{sl}_{2n}\mathbb{C}.$$

(b) The inclusion $\mathfrak{sl}_n\mathbb{C} \times \mathfrak{sl}_m\mathbb{C} \subset \mathfrak{sl}_{n+m}\mathbb{C}$ determines a restriction homomorphism $R(\mathfrak{sl}_{n+m}\mathbb{C}) \to R(\mathfrak{sl}_n\mathbb{C} \times \mathfrak{sl}_m\mathbb{C}) = R(\mathfrak{sl}_n\mathbb{C}) \otimes R(\mathfrak{sl}_m\mathbb{C})$, which takes polynomial generators A_k to $A_k \otimes 1 + A_{k-1} \otimes A_1 + \cdots + 1 \otimes A_k$. Compute analogously for

$$\mathfrak{sp}_{2n}\mathbb{C} \times \mathfrak{sp}_{2m}\mathbb{C} \subset \mathfrak{sp}_{2n+2m}\mathbb{C}, \qquad \mathfrak{so}_n\mathbb{C} \times \mathfrak{so}_m\mathbb{C} \subset \mathfrak{so}_{n+m}\mathbb{C}.$$

Which of these inclusions correspond to removing nodes from the Dynkin diagrams?

Exercise 23.44. Compute the isomorphisms of representation rings corresponding to the isomorphisms $\mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C}$, $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$, and $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$.

§23.3. Homogeneous Spaces

In this section we will introduce and describe the compact homogeneous spaces associated to the classical groups. As we will see, these are classified neatly in terms of Dynkin diagrams, and are, in turn, closely related to the representation theory of the groups acting on them. Unfortunately, we are unable to give here more than the barest outline of this beautiful subject; but we will at least try to say what the principal objects are, and what connections among them exist. In particular, we give at the end of the section a diagram (23.58) depicting these objects and correspondences to which the reader can refer while reading this section.

We begin by introducing the notion of Borel subalgebras and Borel subgroups. Recall first that a choice of Cartan subalgebra $\mathfrak{h}$ in a semisimple Lie algebra $\mathfrak{g}$ determines, as we have seen, a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. To each choice of ordering of the root system $R = R^+ \cup R^-$, we can associate a subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha,$$

called a *Borel subalgebra*. Note that $\mathfrak{b}$ is solvable, since $\mathscr{D}\mathfrak{b} \subset \bigoplus \mathfrak{g}_\alpha$, $\mathscr{D}^2\mathfrak{b} \subset \bigoplus \mathfrak{g}_{\alpha+\beta}$, etc. In fact, $\mathfrak{b}$ is a maximal solvable subalgebra (Exercise 14.35).

If G is a Lie group with semisimple Lie algebra $\mathfrak{g}$, the connected subgroup B of G with Lie algebra $\mathfrak{b}$ is called a *Borel subgroup.*

Claim 23.45. *B is a closed subgroup of G, and the quotient G/B is compact.*

PROOF. Consider the adjoint representation of G on $\mathfrak{g}$. The action of the Borel subalgebra $\mathfrak{b}$ obviously preserves the subspace $\mathfrak{b} \subset \mathfrak{g}$, and, in fact, $\mathfrak{b}$ is just the inverse image of the subalgebra of $\mathfrak{gl}(\mathfrak{g})$ preserving this subspace: if $X = \sum X_\alpha$ is any element of $\mathfrak{g}$ with $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\alpha \neq 0$ for some $\alpha \in R^-$, we could find an element H of $\mathfrak{h} \subset \mathfrak{b}$ with $\mathrm{ad}(X)(H) \notin \mathfrak{b}$—any H not in the annihilator of $\alpha \in \mathfrak{h}^*$ would do. B is thus (the connected component of the identity in) the inverse image in G of the subgroup of $\mathrm{GL}(\mathfrak{g})$ carrying $\mathfrak{b}$ into itself. It follows that B is closed; and the quotient G/B is contained in a Grassmannian and hence compact. (Alternatively, we could consider the action of G on the projective space $\mathbb{P}(\bigwedge^m \mathfrak{g})$, where m is the number of positive roots, and observe that B is the stabilizer of the point corresponding to the exterior product of the positive root spaces.)

In fact, in the case of the classical groups, it is easy to describe the Borel subgroups and the corresponding quotients.

For $G = \mathrm{SL}_{n+1}\mathbb{C}$, B is the group of all upper-triangular matrices in G, i.e., those automorphisms preserving the standard flag. It follows that G/B is the usual (complete) flag manifold, i.e., the variety of all flags

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}\}$$

of subspaces with $\dim(V_r) = r$.

For $G = \mathrm{SO}_{2n+1}\mathbb{C}$ the orthogonal group of automorphisms of $\mathbb{C}^{2n+1}$ preserving a quadratic form Q, B is the subgroup of automorphisms which preserve a fixed flag $V_1 \subset \cdots \subset V_n$ of isotropic subspaces with $\dim(V_r) = r$. All such flags being conjugate, G/B is the variety of all such flags, i.e.,

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{2n+1} \colon Q(V_n, V_n) \equiv 0\}.$$

Note that B automatically preserves the flag of orthogonal subspaces, so that we could also characterize G/B as the space of complete flags equal to their orthogonal complements, i.e.,

$$G/B = \{V_1 \subset \cdots \subset V_{2n} \subset \mathbb{C}^{2n+1} \colon Q(V_i, V_{2n+1-i}) = 0\}.$$

The same holds for $\mathrm{Sp}_{2n}\mathbb{C}$: the Borel subgroups $B \subset \mathrm{Sp}_{2n}\mathbb{C}$ are just the subgroups preserving a half-flag of isotropic subspaces, or equivalently a full flag of pairwise complementary subspaces; and the quotient G/B is correspondingly the variety of all such flags.

For $G = \mathrm{SO}_{2n}\mathbb{C}$, B fixes an isotropic flag $V_1 \subset \cdots \subset V_{n-1}$, and

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^{2n} \colon Q(V_{n-1}, V_{n-1}) = 0\}.$$

Exercise 23.46. With our choice of basis $\{e_i\}$, let V_r be the subspace spanned by the first r basic vectors. If B is defined to be the subgroup that preserves V_r for $1 \leq r \leq n$, verify that the Lie algebra of B is spanned by the Cartan subalgebra and the positive root spaces described in Lectures 17 and 19.

We now want to consider more general quotients of a semisimple complex group G. To begin with, we say that a (closed, complex analytic, and connected[1]) subgroup P of G is *parabolic* if the quotient G/P can be realized as the orbit of the action of G on $\mathbb{P}(V)$ for some representation V of G. In particular, G/P is a projective algebraic variety. It follows from the proof of Claim 23.45 that any Borel subgroup B of G is parabolic. The following two claims characterize parabolic subgroups as those containing a Borel subgroup, i.e., the Borel subgroups are exactly the *minimal* parabolic subgroups.

Claim 23.47. *If B is a Borel subgroup and P a parabolic subgroup of G, then there is an $x \in G$ with*

$$B \subset xPx^{-1}.$$

Claim 23.48. *If a subgroup P of G contains a Borel subgroup B, then P is parabolic.*

The first claim is deduced from a version of *Borel's fixed point theorem*: if B is a connected solvable group, V a representation of B and $X \subset \mathbb{P}V$ a projective variety carried into itself under the action of B on $\mathbb{P}V$, then B must have a fixed point on X. This is straightforward: we observe (by Lie's theorem (9.11)) that the action of the solvable group B on V must preserve a flag of subspaces

$$0 \subset V_1 \subset \cdots \subset V_n = V$$

with $\dim(V_i) = i$. We can thus find a subspace $V_i \subset V$ fixed by B such that X intersects $\mathbb{P}V_i$ in a finite collection of points, which must then be fixed points for the action of B on X. As for Claim 23.48 we will soon see directly how G/P is a projective variety whenever P is a subgroup containing B.

We can now completely classify the parabolic subgroups of a simple group, up to conjugacy. By the above, we may assume that P contains a Borel subgroup B. Correspondingly, its Lie algebra $\mathfrak{p}$ is a subspace of $\mathfrak{g}$ containing $\mathfrak{b}$ and invariant under the action of B on $\mathfrak{g}$; i.e., it is a direct sum

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in T} \mathfrak{g}_\alpha$$

for some subset T of R that contains all positive roots. Now, in order for $\mathfrak{p}$ to be a subalgebra of $\mathfrak{g}$, the subset T must be closed under addition (that is, if two roots are in T, then either their sum is in T or is not a root). Since, in addition, T contains all the positive roots, we may observe that if α, β, and γ are positive roots with $\alpha = \beta + \gamma$, then we must have

$$-\alpha \in T \Rightarrow -\beta \in T \text{ and } -\gamma \in T.$$

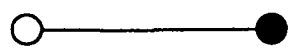
[1] It is a general fact that P must be connected if G/P is a projective variety.

Clearly, any such subset T must be generated by R^+ together with the negatives of a subset Σ of the set of simple roots. Thus, if for each subset Σ of the set of simple roots we let $T(\Sigma)$ consist of all roots which can be written as sums of negatives of the roots in Σ, together with all positive roots, and form the subalgebra

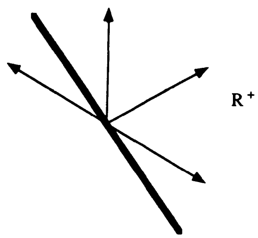
$$\mathfrak{p}(\Sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha \in T(\Sigma)} \mathfrak{g}_\alpha, \tag{23.49}$$

then $\mathfrak{p}(\Sigma)$ is a parabolic subalgebra, the corresponding Lie group $P(\Sigma)$ is a parabolic subgroup containing B, and we obtain in this way all the parabolic subgroups of G. We can express this as the observation that, up to conjugacy, *parabolic subgroups of the simple group G are in one-to-one correspondence with subsets of the nodes of the Dynkin diagram, i.e., with subsets of the set of simple roots.*

Examples. In the case of $\mathfrak{sl}_3\mathbb{C}$, there is a symmetry in the Dynkin diagram, so that there is only one parabolic subgroup other than the Borel, corresponding to the diagram



This, in turn, gives the subset of the root system



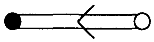
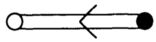
corresponding to the subgroup

$$P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

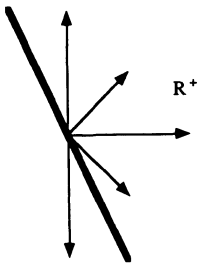
and the homogeneous space

$$G/P = \mathbb{P}^2.$$

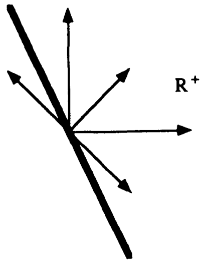
In the case of $\mathfrak{sp}_4\mathbb{C}$, there are two subdiagrams of the Dynkin diagram:

 and 

these correspond to the subsets of the root system



and

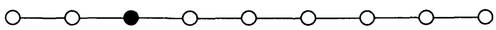


(Here we are using a black dot to indicate an omitted simple root, a white dot to indicate an included one.) The corresponding subgroups of $\mathrm{Sp}_4\mathbb{C}$ are those preserving the vector e_1, and preserving the subspace spanned by e_1 and e_2, respectively. The quotients G/B are thus the variety of one-dimensional isotropic subspaces (i.e., the variety $\mathbb{P}^3$ of all the one-dimensional spaces) and the variety of two-dimensional isotropic subspaces.

Exercise 23.50. Interpret the diagrams above as giving rise to parabolic subgroups of the group $\mathrm{SO}_5\mathbb{C}$ of automorphisms of $\mathbb{C}^5$ preserving a symmetric bilinear form. Show that the corresponding homogeneous spaces are the variety of isotropic planes and lines in $\mathbb{C}^5$, respectively. In particular, deduce the classical algebraic geometry facts that:

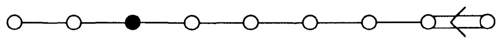
(i) The variety of isotropic 2-planes for a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^4$ is isomorphic to a quadric hypersurface in $\mathbb{P}^4$.
(ii) The variety of isotropic 2-planes for a nondegenerate symmetric bilinear form on $\mathbb{C}^5$ (equivalently, lines on a smooth quadric hypersurface in $\mathbb{P}^4$) is isomorphic to $\mathbb{P}^3$.

In general, it is not hard to see that any parabolic subgroup P in a classical group G may be described as the subgroup that preserves a partial flag in the standard representation. In particular, a maximal parabolic subgroup, corresponding to omitting one node of the Dynkin diagram, may be described as the subgroup of G preserving a single subspace. Thus, for $G = \mathrm{SL}_m\mathbb{C}$, the kth node of the Dynkin diagram

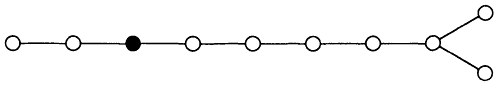


corresponds to the Grassmannian $G(k, m)$ of k-dimensional subspaces of $\mathbb{C}^m$. (Note that the symmetry of the diagram reflects the isomorphism of the Grassmannians $G(k, m)$ and $G(m - k, m)$.)

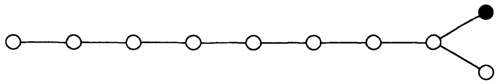
For $\mathrm{Sp}_{2n}\mathbb{C}$, the kth node of the Dynkin diagram



corresponds to the *Lagrangian Grassmannian* of isotropic k-planes, for $k = 1, 2, \ldots, n$. Similarly, for $G = SO_{2n+1}\mathbb{C}$, the kth node of the Dynkin diagram corresponds to the *orthogonal Grassmannian* of isotropic k-planes in $\mathbb{C}^{2n+1}$. Finally, for $SO_{2n}\mathbb{C}$, for $k = 1, 2, \ldots, n-2$ the kth node of the Dynkin diagram



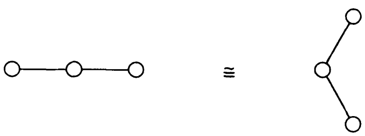
yields the orthogonal Grassmannian of isotropic k-planes in $\mathbb{C}^{2n}$, but there is one anomaly: either of the last two nodes



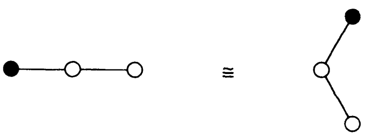
gives one of the two connected components of the Grassmannian of isotropic n-planes.

Exercise 23.51*. Compute $\mathfrak{p}(\Sigma)$ directly for each of the classical groups, and verify the above statements. Why is the orthogonal Grassmannian of isotropic $(n-1)$-planes in $\mathbb{C}^{2n}$ not included on the list?

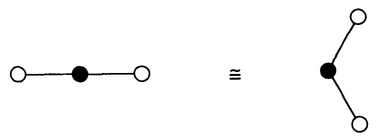
As we saw already in Exercise 23.50, the low-dimensional coincidences between Dynkin diagrams can be used to recover some facts we have seen before. For example, the coincidence $(D_2) = (A_1) \times (A_1)$ identifies the two family of lines on a quadratic surface in $\mathbb{P}^3$ with two copies of $\mathbb{P}^1$. The coincidence $(A_3) = (D_3)$



gives rise to two identifications of marked diagrams: we have



corresponding to the isomorphism between the Grassmann varieties $\mathbb{P}^3 = G(1, 4)$, $\check{\mathbb{P}}^3 = G(3, 4)$ and the two components of the family of 2-planes on a quadric hypersurface Q in $\mathbb{P}^5$; and



corresponding to the isomorphism of the Grassmannian $G(2, 4)$ with the quadric hypersurface Q itself. Finally, an observation that is not quite so elementary, but which we saw in §20.3: the identification of the diagrams

says that *either connected component of the variety of* 3*-planes on a smooth quadric hypersurface Q in $\mathbb{P}^7$ is isomorphic to the quadric Q itself.*

There is another way to realize the compact homogeneous spaces associated to a simple group G. Let $V = \Gamma_\lambda$ be an irreducible representation of G with highest weight λ, and consider the action of G on the projective space $\mathbb{P}V$. Let $p \in \mathbb{P}V$ be the point corresponding to the eigenspace with eigenvalue λ. We have then

Claim 23.52. *The orbit $G \cdot p$ is the unique closed orbit of the action of G on $\mathbb{P}V$.*

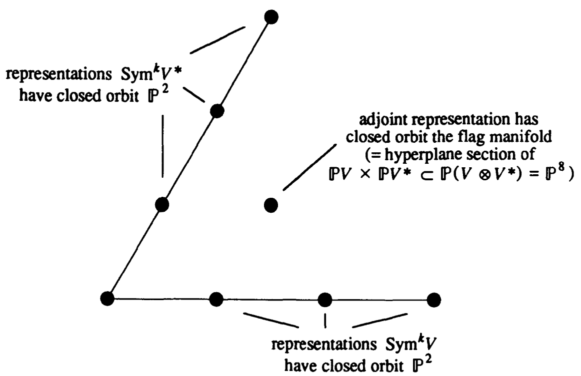
Proof. The point p is fixed under the Borel subgroup B, so that the stabilizer of p is a parabolic subgroup P_λ; the orbit G/P_λ is thus compact and hence closed. Conversely, by the Borel fixed point theorem, any closed orbit of G contains a fixed point for the action of B; but p is the unique point in $\mathbb{P}V$ fixed by B. □

In fact, it is not hard to say which parabolic subgroup P_λ is, in terms of the classification above: *it is the parabolic subgroup corresponding to the subset of simple roots that are perpendicular to the weight λ.* Now, sets Σ of simple roots correspond to faces of the Weyl chamber, namely, the face that is the intersection of all hyperplanes perpendicular to all roots in Σ.

We thus have a correspondence between faces of the Weyl chamber and parabolic subgroups P, such that if $V = \Gamma_\lambda$ is the irreducible representation with highest weight λ, then the unique closed orbit of the action of G on $\mathbb{P}V$

is of the form G/P, where P is the parabolic subgroup corresponding to the open face of $\mathscr{W}$ containing λ. In particular, weights in the interior of the Weyl chamber correspond to $P_\lambda = B$, and so determine the full flag manifold G/B, whereas weights on the edges give rise to the quotients of G by maximal parabolics. Note that we do obtain in this way all compact homogeneous spaces for G.

For example, we have the representations of $\mathrm{SL}_3\mathbb{C}$: as we have seen, the representations $\mathrm{Sym}^k V$ and $\mathrm{Sym}^k V^*$, with highest weights on the boundaries of the Weyl chamber, have closed orbits $\{v^k\}_{v \in V}$ and $\{l^k\}_{l \in V^*}$, isomorphic to $\mathbb{P}V$ and $\mathbb{P}V^*$. By contrast, the adjoint representation—the complement of the trivial representation in $\mathrm{Hom}(V, V) = V \otimes V^*$—has as closed orbit the variety of traceless rank 1 homomorphisms, which is isomorphic to the flag manifold via the map sending a homomorphism φ to the pair $(\mathrm{Im}\ \varphi, \mathrm{Ker}\ \varphi)$. The picture is

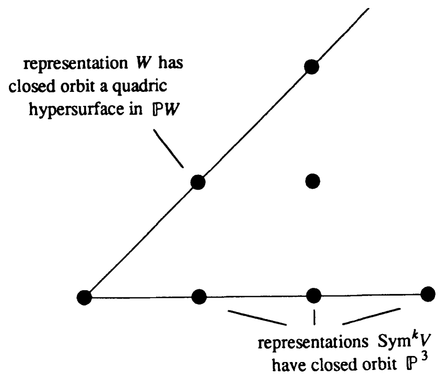


In general, if V is the standard representation of $\mathrm{SL}_n\mathbb{C}$, in the representations of $\mathrm{SL}_n\mathbb{C}$ of the form $W = \mathrm{Sym}^k V$ we saw that the vectors of the form $\{v^k\}_{v \in V}$ formed a closed orbit in $\mathbb{P}W$, called the Veronese embedding of $\mathbb{P}^{n-1}$. Likewise, in representations of the form $W = \bigwedge^k V$ the decomposable vectors $\{v_1 \wedge v_2 \wedge \cdots \wedge v_k\}$ formed a closed orbit in $\mathbb{P}W$; this is the Plücker embedding of the Grassmannian.

Similarly, we may identify the closed orbits in representations of $\mathrm{Sp}_4\mathbb{C}$. Recall here that the basic representations of $\mathrm{Sp}_4\mathbb{C}$ are the standard representation $V \cong \mathbb{C}^4$ and the complement W of the trivial representation in the exterior square $\bigwedge^2 V$; all other representations are contained in a tensor product of symmetric powers of these. Now, $\mathrm{Sp}_4\mathbb{C}$ acts transitively on $\mathbb{P}V$; the closed orbit is all of $\mathbb{P}^3$. In general, in $\mathbb{P}(\mathrm{Sym}^k V)$ the closed orbit is just the set of vectors $\{v^k\}_{v \in V} \cong \mathbb{P}^3$. By contrast, the closed orbit in $\mathbb{P}W$ is just the intersection of the hyperplane $\mathbb{P}W \subset \mathbb{P}(\bigwedge^2 V)$ with the locus of decomposable vectors $\{v \wedge w\}_{v, w \in V}$; this is the variety

$$X = \{v \wedge w: Q(v, w) = 0\}$$

of isotropic 2-planes $\Lambda \subset V$ for the skew form Q.



For the group $\mathrm{Spin}_{2n+1}\mathbb{C}$, the closed orbit of the spin representation S is the orthogonal Grassmannian of n-dimensional isotropic subspaces of $\mathbb{C}^{2n+1}$. The corresponding subvariety

$$G/P \hookrightarrow \mathbb{P}(S)$$

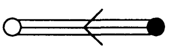
is a variety of dimension $(n+1)n/2$ in $\mathbb{P}^N$, $N = 2^n - 1$, called the *spinor variety*, or the variety of *pure spinors*. Similarly for $\mathrm{Spin}_{2n}\mathbb{C}$, the two spin representations S^+ and S^- give embeddings of the two components of the orthogonal Grassmannian of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$, one in $\mathbb{P}(S^+)$, one in $\mathbb{P}(S^-)$. These spinor varieties have dimension $n(n-1)/2$ in projective spaces of dimension $2^{n-1} - 1$.

Exercise 23.53. Show that the spinor variety for $\mathrm{Spin}_{2n-1}\mathbb{C}$ is isomorphic to each of the spinor varieties for $\mathrm{Spin}_{2n}\mathbb{C}$. In fact they are projectively equivalent as subvarieties of projective space $\mathbb{P}^N$, $N = 2^{n-1} - 1$.

It follows that, for $m \leq 8$, the spinor varieties for $\mathrm{Spin}_m\mathbb{C}$ are isomorphic to homogeneous spaces we have described by other means. The first new one is the 10-dimensional variety in $\mathbb{P}^{15}$, which comes from $\mathrm{Spin}_9\mathbb{C}$ or $\mathrm{Spin}_{10}\mathbb{C}$.

It is worth going back to interpret some of the "geometric plethysm" of earlier lectures (e.g., Exercises 11.36 and 13.24) in this light.

Finally, we can describe (at least one of) the compact homogeneous spaces for the group G_2 in this way. To begin with, G_2 has two maximal parabolic subgroups, corresponding to the diagrams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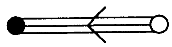
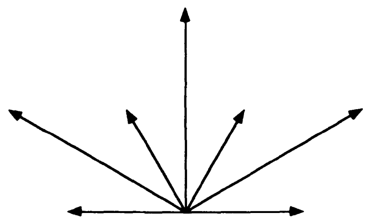


These are the groups whose Lie algebras are the parabolic subalgebras spanned by the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ together with the root spaces corresponding to the roots in the diagrams



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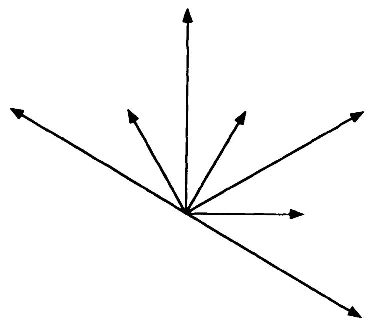
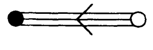


In particular, each of these parabolic subgroups will have dimension 9, so that both the corresponding homogeneous spaces will be five-dimensional varieties. We can use this to identify one of these spaces: if V is the standard seven-dimensional representation of G_2, the closed orbit in $\mathbb{P}V \cong \mathbb{P}^6$ will be a hypersurface, which (since it is homogeneous) can only be a quadric hypersurface. Thus, the homogeneous space for G_2 corresponding to the diagram



is a quadric hypersurface in $\mathbb{P}^6$. In particular, we see again that the action of G_2 on V preserves a nondegenerate bilinear form, i.e., we have an inclusion

$$G_2 \hookrightarrow \mathrm{SO}_7\mathbb{C}.$$

The other homogeneous space Y of $\mathfrak{g}_2$ is less readily described. One way to describe it is to use the fact that the adjoint representation W of $\mathfrak{g}_2$ is contained in the exterior square $\wedge^2 V$ of the standard. Since the Grassmannian $\mathbb{G}(1, 7) \subset \mathbb{P}(\wedge^2 V)$ of lines in $\mathbb{P}V$ is closed and invariant in $\mathbb{P}(\wedge^2 V)$, it follows that Y is contained in the intersection of $\mathbb{G}$ with the subspace $\mathbb{P}W \subset \mathbb{P}(\bigwedge^2 V)$. In other words, in terms of the skew-symmetric trilinear form ω on V preserved by the action of G_2, we can say that Y is contained in the locus

$$\Sigma = \{\Lambda \subset V: \omega(\Lambda, \Lambda, \cdot) \equiv 0\} \subset G(2, V).$$

Problem 23.54. Is $Y = \Sigma$?

Exercise 23.55. Show that the representation of E_6 whose highest weight is the first fundamental weight ω_1 determines a 16-dimensional homogeneous space in $\mathbb{P}^{26}$.

These homogeneous spaces have an amazing way of showing up as extremal examples of subvarieties of projective spaces, starting with a discovery of Severi that the Veronese surface in $\mathbb{P}^5$ is the only surface in $\mathbb{P}^5$ (nonsingular and not contained in a hyperplane) whose chords do not fill up $\mathbb{P}^5$. For recent work along these lines, see [L-VdV], with its appendix by Zak on interesting projective varieties that arise from representation theory.

Although we have described homogeneous spaces only for semisimple Lie groups, this is no real loss of generality: any irreducible representation V of a Lie group G comes from a representation of its semisimple quotient, up to multiplying by a character (see Proposition 9.17), and this character does not change the orbits in $\mathbb{P}(V)$.

It is possible to take this whole correspondence one step further and use it to give a construction of the irreducible representations of G; this is the modern approach to constructing the irreducible representations, due primarily to Borel, Weil, Bott, and, in a more general setting, Schmid. We do not have the means to do this in detail in the present circumstances, but we will sketch the construction.

The idea is very straightforward. We have just seen that for every irreducible representation V of G there is a unique closed orbit $X = G/P$ of the action of G on $\mathbb{P}V$. We obtain in this way from V a projective variety X together with a line bundle L on X invariant under the action of G (the restriction of the universal bundle from $\mathbb{P}V$). In fact, we may recover V from this data simply as the vector space of holomorphic sections of the line bundle L on X. What ties this all together is the fact that this gives us a one-to-one correspondence between irreducible representations of G and ample (positive) line bundles on compact homogeneous spaces G/P. More generally, using the projection maps $G/B \to G/P$, we may pull back all these line bundles to line bundles on G/B. This then extends to give an isomorphism between the weight lattice of $\mathfrak{g}$ and the group of line bundles on G/B, with the wonderful property that for dominant weights λ, the space of holomorphic sections of the associated line bundle L_λ is the irreducible representation of G with highest weight λ.

The point of all this, apart from its intrinsic beauty, is that we can go backward: starting with just the group G, we can construct the homogeneous space G/B, and then realize all the irreducible representations of G as cohomology groups of line bundles on G/B. To carry this out, start with a weight $\lambda \in \mathfrak{h}^*$ for $\mathfrak{g}$. We have seen that λ exponentiates to a homomorphism $H \to \mathbb{C}^*$, i.e., it gives a one-dimensional representation $\mathbb{C}_\lambda$ of H. We want to induce this representation from H to G. If $H \subset B \subset G$ is a Borel subgroup, the representation extends trivially to B, since B is a semidirect product of H and the nilpotent subgroup N whose Lie algebra is the direct sum of those $\mathfrak{g}_\alpha$ for positive roots α. Then we can form

$$\begin{aligned} L_\lambda &= G \times_B \mathbb{C}_\lambda \\ &= (G \times \mathbb{C}_\lambda)/\{(g, v) \sim (gx, x^{-1}v), x \in B\}, \end{aligned}$$

which, with its natural projection to G/B, is a holomorphic line bundle on the projective variety G/B. The cohomology groups of such a line bundle are finite dimensional, and since G acts on L_λ, these cohomology groups are representations of G.

We have Bott's theorem for the vanishing of the cohomology of this line bundle:

Claim 23.56. $H^i(G/B, L_\lambda) = 0$ *for* $i \neq i(\lambda)$,

where $i(\lambda)$ is an integer depending on which Weyl chamber λ belongs to. If λ is a dominant weight (i.e., belongs to the closure of the positive Weyl chamber for the choice of positive roots used in defining B), then $i(-\lambda) = 0$. In this case the sections $H^0(G/B, L_{-\lambda})$ are a finite-dimensional vector space, on which G acts.

Claim 23.57. *For λ a dominant weight, the space of sections $H^0(G/B, L_{-\lambda})$ is the irreducible representation with highest weight λ.*

In this context the Riemann–Roch theorem can be applied to give a formula for the dimension of the irreducible representation. In fact, the dimension part of Weyl's character formula can be proved this way. More refined analysis, using the Woods Hole fixed point theorem, can be used to get the full character formula (cf. [A-B]). For a very readable introduction to this, see [Bot].

We conclude this discussion by giving a diagram showing the relationships among the various objects associated to an irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. The objects and maps in diagram (23.58) are explained next.

First of all, as we have indicated, the term "Grassmannians" means the ordinary Grassmannians in the case of the groups $SL_n\mathbb{C}$, and the Lagrangian Grassmannians and the orthogonal Grassmannians of isotropic subspaces in the cases of $Sp_{2n}\mathbb{C}$ and $SO_m\mathbb{C}$, respectively. Likewise, "flag manifolds" refers to the spaces parametrizing nested sequences of such subspaces. In the cases

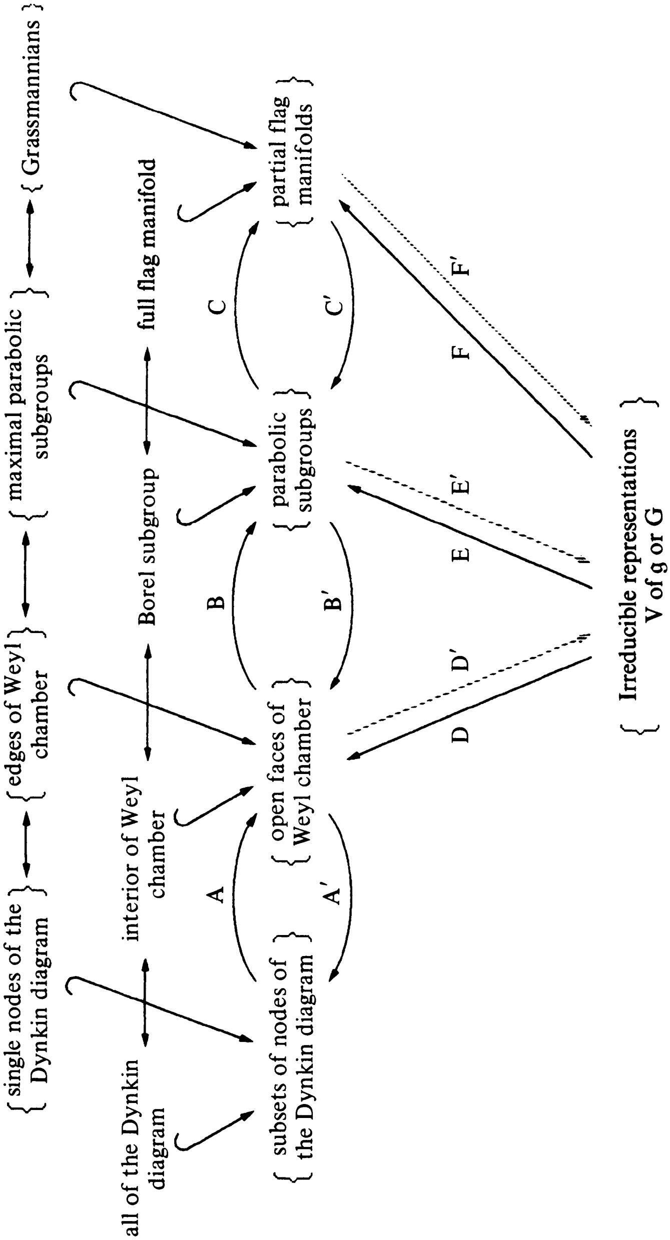


Diagram 23.58

of the exceptional Lie algebras, the term "Grassmannian" should just be ignored; except for the quotient of G_2 by one of its two maximal parabolic subgroups, the homogeneous spaces for the exceptional groups are not varieties with which we are likely to be a priori familiar.

With this said, we may describe the maps A, B, etc., as follows:

A, A': the map A associates to a subset of the nodes of the Dynkin diagram (equivalently, a subset S of the set of simple roots) the face of the Weyl chamber described by

$$\mathscr{W}_S = \left\{\lambda: \begin{matrix}(\lambda, \alpha) > 0, \forall \alpha \in S; \\ (\lambda, \alpha) = 0, \forall \alpha \notin S\end{matrix}\right\},$$

where (,) is the Killing form; the inverse is clear.

B, B': the map B associates to a face $\mathscr{W}_S$ of the Weyl chamber the subalgebra $\mathfrak{g}_S$ spanned by the Cartan subalgebra $\mathfrak{h}$, the positive root spaces $\mathfrak{g}_\alpha$, $\alpha \in R^+$, and the root spaces $\mathfrak{g}_{-\alpha}$ corresponding to those positive roots α perpendicular to $\mathscr{W}_S$. Equivalently, in terms of the corresponding subset S of the simple roots, $\mathfrak{g}_S$ will be generated by the Borel subalgebra, together with the root spaces $\mathfrak{g}_{-\alpha}$ for $\alpha \notin S$. Again, since every parabolic subalgebra is conjugate to one of this form, the inverse map is clear.

C, C': The map C simply associates to a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ the quotient G/P of G by the corresponding parabolic subgroup $P \subset G$. In the other direction, given the homogeneous space $X = G/P$, with the action of G, the group P is just the stabilizer of a point in X. Note that the connected component of the identity in the automorphism group of G/P may be strictly larger: for example, $\mathbb{P}^{2n-1}$ is a compact homogeneous space for $Sp_{2n}\mathbb{C}$, and we have seen that a quadric hypersurface in $\mathbb{P}^6$ is a homogeneous space for G_2.

D, D': The map D associates to the irreducible representation V of $\mathfrak{g}$ with highest weight λ the open face of the Weyl chamber containing λ. In the other direction, given an open face $\mathscr{W}_S$ of $\mathscr{W}$, choose a lattice point $\lambda \in \mathscr{W}_S \cap \Lambda_W$ and take $V = \Gamma_\lambda$.

E: We send the representation V to the subalgebra or subgroup fixing the highest weight vector $v \in V$.

F, F': We associate to the representation V the (unique) closed orbit of the corresponding action of the group G on the projective space $\mathbb{P}V$. Going in the other direction, we have to choose an ample line bundle L on the space G/P, and then take its vector space of holomorphic sections.

§23.4. Bruhat Decompositions

We end this lecture with a brief introduction to the *Bruhat decomposition* of a semisimple complex Lie group G, and the related *Bruhat cells* in the flag manifold G/B. These ideas are not used in this course, but they appear so often elsewhere that it may be useful to describe them in the language we have developed in this lecture. We will give the general statements, but verify them only for the classical groups. General proofs can be found in [Bor1] or [Hu2].

As we have seen, a choice of positive roots determines a Borel subgroup B and Cartan subgroup H, with normalizer $N(H)$, so $N(H)/H$ is identified with the Weyl group $\mathfrak{W}$. For each $W \in \mathfrak{W}$ fix a representative n_W in $N(H)$. The double coset $B \cdot n_W \cdot B$ is clearly independent of choice of n_W, and will be denoted $B \cdot W \cdot B$.

Theorem 23.59 (Bruhat Decomposition). *The group G is a disjoint union of the $|\mathfrak{W}|$ double cosets $B \cdot W \cdot B$, as W varies over the Weyl group.*

Let us first see this explicitly for $G = \mathrm{SL}_m\mathbb{C}$. Here $N(H)$ consists of all monomial matrices in $\mathrm{SL}_m\mathbb{C}$, i.e., matrices with exactly one nonzero entry in each row and each column, and $\mathfrak{W} = \mathfrak{S}_m$; a monomial matrix with nonzero entry in the $\sigma(j)$th row of the jth column maps to the permutation σ. To see that the double cosets cover G, given $g \in G$, use elementary row operations by left multiplication by elements in B to get an element $b \cdot g^{-1}$, with $b \in B$ chosen so that the total number of zeros appearing at the left in the rows in $b \cdot g^{-1}$ is as large as possible. If two rows of $b \cdot g^{-1}$ had the same number of zeros at the left, one could increase the total by an elementary row operation. Since all the rows of $b \cdot g^{-1}$ start with different numbers of zeros, this matrix can be put in upper-triangular form by left multiplication by a monomial matrix; therefore, there is a permutation σ so that $b' = n_\sigma \cdot b \cdot g^{-1}$ is upper triangular, i.e., $g = (b')^{-1} \cdot n_\sigma \cdot b$ is in $B \cdot \sigma \cdot B$. To see that the double cosets are disjoint, suppose $n_{\sigma'} = b' \cdot n_\sigma \cdot b$ for some b and b' in B. From the equation $b = (n_\sigma)^{-1} \cdot (b')^{-1} \cdot n_{\sigma'}$ one sees that b must have nonzero entries in each place where $(n_\sigma)^{-1} \cdot n_{\sigma'}$ does, from which it follows that $\sigma' = \sigma$.

In fact, this can be strengthened as follows. Let U (resp. U^-) be the subgroup of G whose Lie algebra is the sum of all root spaces $\mathfrak{g}_\alpha$ for all positive (resp. negative) roots α. For $G = \mathrm{SL}_m\mathbb{C}$, U (resp. U^-) consists of upper- (resp. lower-) triangular matrices with 1's on the diagonal. For W in the Weyl group, define subgroups

$$U(W) = U \cap n_W \cdot U^- \cdot n_W^{-1}, \qquad U(W)' = U \cap n_W \cdot U \cdot n_W^{-1}$$

of U, which are again independent of the choice of representative n_W for W.

Corollary 23.60. *Every element in $B \cdot W \cdot B$ can be written $u \cdot n_W \cdot b$ for unique elements u in $U(W)$ and b in B.*

To see the existence of such an expression, note first that the Lie algebra of $U(W)$ is the sum of all root spaces $\mathfrak{g}_\alpha$ for which α is positive and $W^{-1}(\alpha)$ is negative; and the Lie algebra of $U(W)'$ is the sum of all root spaces $\mathfrak{g}_\alpha$ for which α and $W^{-1}(\alpha)$ are positive. One sees from this that $U(W) \cdot U(W)' \cdot H$ is the entire Borel group B. Since $H \cdot n_W = n_W \cdot H$ and $U(W)' \cdot n_W = n_W \cdot U$, and H and U are subgroups of B,

$$\begin{aligned} B\cdot n_W\cdot B &= U(W)\cdot U(W)'\cdot H\cdot n_W\cdot B \\ &= U(W)\cdot U(W)'\cdot n_W\cdot B \\ &= U(W)\cdot n_W\cdot B. \end{aligned}$$

To see the uniqueness, suppose that $n_W = u\cdot n_W\cdot b$ for some u in $U(W)$ and b in B. Then $n_W^{-1}\cdot u\cdot n_W$ is in $U^- \cap B = \{1\}$, so $u = 1$, as required.

Note in particular that the dimension of $U(W)$ is the cardinality of $R^+ \cap W(R^-)$, where R^+ and R^- are the positive and negative roots; this is also the minimum number $l(W)$ of reflections in simple roots whose product is W, cf. Exercise D.30. It is a general fact, which we will see for the classical groups, that $U(W)$ is isomorphic to an affine space $\mathbb{C}^{l(W)}$.

It follows from the Bruhat decomposition that G/B is a disjoint union of the cosets $X_W = B\cdot n_W\cdot B/B$, again with W varying over the Weyl group. These X_W are called *Bruhat cells*. From the corollary we see that X_W is isomorphic to the affine space $U(W) \cong \mathbb{C}^{l(W)}$.

For $G = \mathrm{SL}_m\mathbb{C}$ and σ in $\mathfrak{W} = \mathfrak{S}_m$, the group $U(\sigma)$ consists of matrices with 1's on the diagonal, and zero entry in the i, j place whenever either $i > j$ or $\sigma^{-1}(i) < \sigma^{-1}(j)$, which is an affine space of dimension $l(\sigma) = \#\{(i, j)\colon i > j \text{ and } \sigma(i) < \sigma(j)\}$.

Exercise 23.61. Identifying $\mathrm{SL}_m\mathbb{C}/B$ with the space of all flags, show that X_σ consists of those flags $0 \subset V_1 \subset V_2 \subset \cdots$ such that the dimensions of intersections with the standard flag are governed by σ, in the following sense: for each $1 \le k \le m$, the set of k numbers d such that $V_k \cap \mathbb{C}^{d-1} \subsetneq V_k \cap \mathbb{C}^d$ is precisely the set $\{\sigma(1), \sigma(2), \ldots, \sigma(k)\}$.

We will verify the Bruhat decomposition for $\mathrm{Sp}_{2n}\mathbb{C}$ by regarding it as a subgroup of $\mathrm{SL}_{2n}\mathbb{C}$ and using what we have just seen for $\mathrm{SL}_{2n}\mathbb{C}$, following [Ste2]. Our description of $\mathrm{Sp}_{2n}\mathbb{C}$ in Lecture 16 amounts to saying that it is the fixed point set of the automorphism φ of $\mathrm{SL}_{2n}\mathbb{C}$ given by $\varphi(A) = M^{-1}\cdot {}^tA^{-1}\cdot M$, with $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The Borel subgroup of $\mathrm{Sp}_{2n}\mathbb{C}$ will be the intersection of the Borel subgroup B of $\mathrm{SL}_{2n}\mathbb{C}$ with $\mathrm{Sp}_{2n}\mathbb{C}$, provided we change the order of the basis of $\mathbb{C}^{2n}$ to $e_1, \ldots, e_n, e_{2n}, \ldots, e_{n+1}$, so that B consists of matrices whose upper left block is upper triangular, whose lower left block is zero, and whose lower right block is lower triangular. The automorphism φ maps this B to itself, and also preserves the diagonal subgroup H and its normalizer $N(H)$, and the groups U and U^-. The Weyl group of $\mathrm{Sp}_{2n}\mathbb{C}$ can be identified with the permutations in $\mathfrak{S}_{2n}$ such that $\sigma(n + i) = \sigma(i) \pm n$ for all $1 \le i \le n$, and it is exactly for these σ for which one can choose a monomial representative n_σ in $\mathrm{Sp}_{2n}\mathbb{C}$. Now if g is any element in $\mathrm{Sp}_{2n}\mathbb{C}$, write $g = u\cdot n_\sigma\cdot b$ according to the above corollary. Then

$$g = \varphi(g) = \varphi(u)\cdot\varphi(n_\sigma)\cdot\varphi(b),$$

and by uniqueness of the decomposition we must have $\varphi(u) = u$, $\varphi(n_\sigma) = n_\sigma \cdot h$, $h \in H$, and $\varphi(b) = h^{-1} \cdot b$. It follows that σ belongs to the Weyl group of $\mathrm{Sp}_{2n}\mathbb{C}$. This gives the Bruhat decomposition, and, moreover, a unique decomposition of $g \in \mathrm{Sp}_{2n}\mathbb{C}$ into $u \cdot n_\sigma \cdot b$, with u in $U(\sigma) \cap \mathrm{Sp}_{2n}\mathbb{C}$. Since this latter is an affine space, this shows that the corresponding Bruhat cell in the symplectic flag manifold is an affine space.

Exactly the same idea works for the orthogonal groups $\mathrm{SO}_m\mathbb{C}$, by realizing them as fixed points of automorphisms of $\mathrm{SL}_m\mathbb{C}$ of the form $A \mapsto M^{-1} \cdot {}^tA^{-1} \cdot M$, with M the matrix giving the quadratic form.

Note finally that if W' is the element in the Weyl group that takes each Weyl chamber to its negative, then $B \cdot W' \cdot B$ is a dense open subset of G, a fact which is evident for the classical groups by the above discussion. The corresponding Bruhat cell $X_{W'}$ is the image of U^- in G/B, which is also a dense open set. It follows that a function or section of a line bundle on G/B is determined by its values on U^-. For treatises developing representation theory via functions on U^-, see [N-S] or [Žel].

The following exercise uses these ideas to sketch a proof of Claim 23.57 that the sections of the bundle $L_{-\lambda}$ on G/B form the irreducible representation with highest weight λ:

Exercise 23.62*. (a) Show that sections s of $L_{-\lambda}$ are all of the form $s(gB) = (g, f(g))$, where f is a holomorphic function on G satisfying

$$f(g \cdot x) = \lambda(x) f(g) \quad \text{for all } x \in B.$$

(b) Let $n' \in N(H)$ be a representative of the element W' in the Weyl group which takes each element to its negative. Show that f is determined by its value at n'.

(c) Show that any highest weight for f must be λ, and conclude that $H^0(G/B, L_{-\lambda})$ is the irreducible representation Γ_λ with highest weight λ.

The holomorphic functions f of this exercise are functions on the space G/U. In other words, all irreducible representations of G can be found in spaces of functions on G/U. This is one common approach to the study of representations, especially by the Soviet school, cf. [N-S], [Žel].

Functions on G/U form a commutative ring, which indicates how to make the sum of all the irreducible representations into a commutative ring. In fact, for the classical groups, these rings are the algebras $\mathbb{S}^{\bullet}$, $\mathbb{S}^{\langle\bullet\rangle}$, and $\mathbb{S}^{[\bullet]}$ constructed in Lectures 15, 17, and 19, cf. [L-T]. They are also coordinate rings for natural embeddings of flag manifolds in products of projective spaces.

LECTURE 24

Weyl Character Formula

This lecture is pretty straightforward: we simply state the Weyl character formula in §24.1, then show how it may be worked out in specific examples in §24.2. In particular, we derive in the case of the classical algebras formulas for the character of a given irreducible representation as a polynomial in the characters of certain basic ones (either the alternating or the symmetric powers of the standard representation for $\mathfrak{sl}_n\mathbb{C}$ and their analogues for $\mathfrak{sp}_{2n}\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$). The proofs of the formula are deferred to the following two lectures. The techniques involved here are elementary, though the determinantal formulas are fairly complex, involving all the algebra of Appendix A.

§24.1. The Weyl Character Formula

We have already seen the Weyl character formula in the case of $\mathfrak{sl}_n\mathbb{C}$, and it is one reason why we were able to calculate so many more representations in that case. We saw in Lectures 6 and 15 that for the representation $\Gamma_\lambda = \mathbb{S}_\lambda\mathbb{C}^n$ of $SL_n\mathbb{C}$ with highest weight $\lambda = \sum \lambda_i L_i$, the trace of the action of a diagonal matrix $A \in SL_n\mathbb{C}$ with entries $x_1, \ldots, x_n$ is the symmetric function called the Schur polynomial $S_\lambda(x_1, \ldots, x_n)$. This included a formula for the multiplicities, which are the coefficients of the monomials in these variables.

In order to extend this formula to the other Lie algebras, let us try to rewrite this Schur polynomial in a way that may generalize. The Schur polynomial is defined to be a quotient of two alternating polynomials:

$$S_\lambda(x_1, \ldots, x_n) = \frac{|x_j^{\lambda_i+n-i}|}{|x_j^{n-i}|}.$$

These determinants can be expanded as usual as a sum over the symmetric group $\mathfrak{S}_n$, which is the Weyl group $\mathfrak{W}$. Writing $x_i = e(L_i)$ in $\mathbb{Z}[\Lambda]$ as in the preceding lecture, and writing $(-1)^W$ for $\operatorname{sgn}(W) = \det(W)$ for W in the Weyl group, the numerator may be expanded in the form

$$\sum_{W \in \mathfrak{W}} (-1)^W x_{W(1)}^{\lambda_1+n-1} \cdot \ldots \cdot x_{W(n)}^{\lambda_n} = \sum_{W \in \mathfrak{W}} (-1)^W e(W(\Sigma(\lambda_i + n - i)L_i))$$
$$= \sum_{W \in \mathfrak{W}} (-1)^W e(W(\lambda + \rho)),$$

where we write λ for $\Sigma\lambda_i L_i$ and we set $\rho = \Sigma(n-i)L_i$. Our formula therefore takes the form

$$\operatorname{Char}(\Gamma_\lambda) = \frac{\sum(-1)^W e(W(\lambda+\rho))}{\Sigma(-1)^W e(W(\rho))}.$$

The denominator is the discriminant

$$\Delta(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j) = \prod_{i<j} (e(L_i) - e(L_j)).$$

This can be written in terms of the positive roots $L_i - L_j$, $i < j$, as

$$\Delta(x_1, \ldots, x_n) = \prod_{i<j} (e(\tfrac{1}{2}(L_i - L_j) - e(-\tfrac{1}{2}(L_i - L_j))).$$

Note also that

$$\rho = \Sigma(n-i)L_i = L_1 + (L_1 + L_2) + \cdots + (L_1 + \cdots + L_{n-1})$$
$$= \frac{1}{2} \sum_{i<j} (L_i - L_j),$$

which is the *sum of the fundamental weights*, and *half the sum of the positive roots.*

These are the formulas that generalize to the other semisimple Lie algebras: For any weight μ, define $A_\mu \in \mathbb{Z}[\Lambda]$ by

$$A_\mu = \sum_{W \in \mathfrak{W}} (-1)^W e(W(\mu)). \tag{24.1}$$

Note that A_μ is not invariant by the Weyl group, but is *alternating*: $W(A_\mu) = (-1)^W A_\mu$ for $W \in \mathfrak{W}$. The ratio of two alternating polynomials will be invariant.

Theorem 24.2 (Weyl Character Formula). *Let ρ be half the sum of the positive roots. Then ρ is a weight, and $A_\rho \neq 0$. The character of the irreducible representation Γ_λ with highest weight λ is*

$$\operatorname{Char}(\Gamma_\lambda) = \frac{A_{\lambda+\rho}}{A_\rho}. \tag{WCF}$$

The assertions about ρ are part of the following lemma and exercise, which will also be useful in the applications:

Lemma 24.3. *The denominator A_ρ of Weyl's formula is*

$$\begin{aligned} A_\rho &= \prod_{\alpha \in R^+} (e(\alpha/2) - e(-\alpha/2)) \\ &= e(\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha)) \\ &= e(-\rho) \prod_{\alpha \in R^+} (e(\alpha) - 1). \end{aligned}$$

Proof. Since $e(\rho) = e(\sum \alpha/2) = \prod e(\alpha/2)$, the equality of the three displayed expressions is evident; denote these expressions temporarily by A. The key point is to see that A is alternating. For this, it suffices to see that A changes sign when a reflection in a hyperplane perpendicular to one of the simple roots is applied to it, since these reflections generate the Weyl group. This follows immediately from the first expression for A and (a) in Exercise 24.4 below.

Now, by the second displayed expression, the highest weight term that appears in A is $e(\rho)$, which is the same as that appearing in A_ρ. Calculating $1/A$ formally as in (24.5) below, we see that A_ρ/A is a formal sum $\sum m_\mu e(\mu)$ that is invariant by the Weyl group, and, using part (c) of the following exercise, it has weight 0. As in Theorem 23.24 it follows that A_ρ/A is constant; and, since A and A_ρ have the same leading term $e(\rho)$, we must have $A_\rho = A$. □

Exercise 24.4*. (a) If $W = W_{\alpha_i}$ is the reflection in the hyperplane perpendicular to a simple root α_i, show that $W(\alpha_i) = -\alpha_i$, and W permutes the other positive roots.

(b) With W as in (a), show that $W(\rho) = \rho - \alpha_i$. Deduce that ρ is the element in $\mathfrak{h}^*$ such that $\rho(H_{\alpha_i}) = 2(\rho, \alpha_i)/(\alpha_i, \alpha_i) = 1$ for each simple root α_i. Equivalently, ρ is the sum of the fundamental weights. In particular, ρ is a weight.

(c) For any $W \neq 1$ in the Weyl group, show that $\rho - W(\rho)$ is a sum of distinct positive roots. Deduce that $W(\rho)$ is not in the closure of the positive Weyl chamber.

Proofs of the character formula will be given in §25.2 and again in §26.2. For now we should at least verify that it is plausible, i.e., that $A_{\lambda+\rho}/A_\rho$ is in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ and that the highest weight that occurs is λ. Note that since the numerator and denominator are alternating, the ratio is invariant. The fact that A_ρ is not zero follows from the second expression in the preceding lemma. To see that the ratio is actually in $\mathbb{Z}[\Lambda]$, however, we must verify that it has only a finite number of nonzero coefficients. Write

$$\frac{1}{A_\rho} = e(-\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha))^{-1} = e(-\rho) \prod_{\alpha} \sum_{n=0}^{\infty} e(-n\alpha). \tag{24.5}$$

When this is multiplied by $A_{\lambda+\rho} = \sum (-1)^W e(W(\lambda + \rho))$, we get a formal sum where the highest weight that occurs is the weight λ. This means in particular that there are only a finite number of nonzero terms corresponding to weights in the fundamental (positive) Weyl chamber $\mathscr{W}$. But since the ratio is invariant by the Weyl group, the same is true for all Weyl chambers, so $A_{\lambda+\rho}/A_\rho$ is in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$, and has highest weight λ. It follows in particular that the $A_{\lambda+\rho}/A_\rho$, as λ varies over $\mathscr{W} \cap \Lambda$, form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$.

Before considering the proof or any other special cases, we apply (WCF) to give a formula for the dimension of Γ_λ:

Corollary 24.6. *The dimension of the irreducible representation Γ_λ is*

$$\dim \Gamma_\lambda = \prod_{\alpha \in R^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \prod_{\alpha \in R^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where $\langle \alpha, \beta \rangle = \alpha(H_\beta) = 2(\alpha, \beta)/(\beta, \beta)$ and $(\ ,\ )$ is the Killing form.

Proof. The dimension of Γ_λ is obtained by adding the coefficients of all $e(\alpha)$ in $\mathrm{Char}(\Gamma_\lambda)$, i.e., computing the image of $\mathrm{Char}(\Gamma_\lambda)$ by the homomorphism from $\mathbb{Z}[\Lambda]$ to $\mathbb{C}$ which sends each $e(\alpha)$ to 1. However, as in the case of the Schur polynomial, the denominator vanishes if we try to do this directly. To get around this, we factor this homomorphism through the ring of power series:

$$\mathbb{Z}[\Lambda] \xrightarrow{\Psi} \mathbb{C}[[t]] \to \mathbb{C},$$

where the second homomorphism sets the variable t equal to zero, i.e., picks off the constant term of the power series, and the first homomorphism Ψ takes $e(\alpha)$ to $e^{(\rho, \alpha)t}$. More generally, for any weight μ define a homomorphism

$$\Psi_\mu \colon \mathbb{Z}[\Lambda] \to \mathbb{C}[[t]], \qquad e(\alpha) \mapsto e^{(\mu, \alpha)t}.$$

We claim that $\Psi_\mu(A_\lambda) = \Psi_\lambda(A_\mu)$ for all λ and μ. This is a simple consequence of the invariance of the metric $(\ ,\ )$ under the Weyl group:

$$\begin{aligned}
\Psi_\mu(A_\lambda) &= \sum (-1)^W e^{(\mu, W(\lambda))t} \\
&= \sum (-1)^W e^{(W^{-1}(\mu), \lambda)t} \\
&= \sum (-1)^W e^{(W(\mu), \lambda)t} \\
&= \Psi_\lambda(A_\mu).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Psi(A_\lambda) &= \Psi_\rho(A_\lambda) = \Psi_\lambda(A_\rho) \\
&= \prod_{\alpha \in R^+} (e^{(\lambda, \alpha)t/2} - e^{-(\lambda, \alpha)t/2})
\end{aligned}$$

$$= \left(\prod_{\alpha \in R^+} (\lambda, \alpha) \right) t^{\#(R^+)} + \text{terms of higher degree in } t.$$

Hence,

$$\begin{aligned} \Psi(A_{\lambda+\rho}/A_\rho) &= \Psi(A_{\lambda+\rho})/\Psi(A_\rho) \\ &= \frac{\prod(\lambda + \rho, \alpha)}{\prod(\rho, \alpha)} + \text{terms of positive degree in } t, \end{aligned}$$

which finishes the proof. □

Exercise 24.7. In the case of $\mathfrak{sl}_n\mathbb{C}$, verify that the above corollary gives the dimension we found in Lecture 6.

Exercise 24.8. Verify directly that the right-hand side of the formula for the dimension is positive.

Since $\chi_\lambda = A_{\lambda+\rho}/A_\rho$ is the character of a virtual representation which takes on a positive value at the identity, as in the case of finite groups, to prove that it is the character of an irreducible representation, it suffices to show that $\int_G \chi_\lambda \overline{\chi}_\lambda = 1$ for an appropriate compact group G. This was the original approach of Weyl, which we will describe in the last lecture. Since the highest weight appearing is λ, we will know then that this irreducible representation must be Γ_λ.

Exercise 24.9. Use Corollary 24.6 to show that if λ is a dominant weight (i.e., in the closure of the positive Weyl chamber), and ω is a fundamental weight, then the dimension of $\Gamma_{\lambda+\omega}$ is greater than the dimension of Γ_λ. Conclude that the nontrivial representations of smallest dimension must be among the n representations Γ_ω with ω a fundamental weight.

§24.2. Applications to Classical Lie Algebras and Groups

In the case of the general linear group $GL_n\mathbb{C}$, the character[1] of the representation Γ_λ is the Schur polynomial

$$S_\lambda(x_1, \ldots, x_n) = \frac{|x_j^{\lambda_i+n-i}|}{|x_j^{n-i}|},$$

[1] We use the representation of $GL_n\mathbb{C}$ instead of its restriction to $SL_n\mathbb{C}$, since the latter would require the product of the variables x_i to be 1.

which has several expressions in terms of simpler symmetric functions. Note that the character of the dth symmetric power of the standard representation is the dth complete symmetric polynomial H_d in n variables (Appendix A.1):

$$H_d = \mathrm{Char}(\mathrm{Sym}^d(\mathbb{C}^n)).$$

The first "Giambelli" or determinantal formula (A.5) of Appendix A gives the character of the representation with highest weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ as an $r \times r$ determinant:

$$\mathrm{Char}(\Gamma_\lambda) = |H_{\lambda_i+j-i}| = \begin{vmatrix} H_{\lambda_1} & H_{\lambda_1+1} \cdots H_{\lambda_1+k-1} \\ H_{\lambda_2-1} H_{\lambda_2} \cdots & \\ \vdots & \\ H_{\lambda_k-k+1} \cdots & H_{\lambda_k} \end{vmatrix}. \tag{24.10}$$

Equivalently, this expresses a general element $\Gamma_\lambda \in R(G)$ of the representation ring as a polynomial in the representations $\mathrm{Sym}^d(\mathbb{C}^n)$. A second determinantal formula, from (A.6), expresses Γ_λ in terms of the basic representations $\bigwedge^d(\mathbb{C}^n)$, whose characters are the elementary symmetric polynomials

$$E_d = \mathrm{Char}(\textstyle\bigwedge^d(\mathbb{C}^n)).$$

This formula is, with μ the conjugate partition to λ,

$$\mathrm{Char}(\Gamma_\lambda) = |E_{\mu_i+j-i}| = \begin{vmatrix} E_{\mu_1} & E_{\mu_1+1} \cdots E_{\mu_1+l-1} \\ E_{\mu_2-1} E_{\mu_2} \cdots & \\ \vdots & \\ E_{\mu_l-l+1} \cdots & E_{\mu_l} \end{vmatrix} \tag{24.11}$$

In this section we work out the character formula for the other classical Lie algebras, including analogues of these determinantal formulas. The analogues of the first determinantal formula (24.10) were given by Weyl, but the analogues of (24.11) were found only recently ([D'H], [Ko-Te]). We also pay, at least by way of exercises, the debts to (WCF) that we owe from earlier lectures.

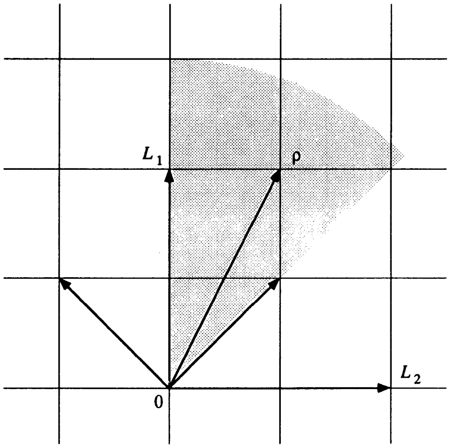
The Symplectic Case

The weights for $\mathfrak{sp}_{2n}\mathbb{C}$ are integral linear combinations of $L_1, \ldots, L_n$. We often write $\mu = (\mu_1, \ldots, \mu_n)$ for the weight $\mu_1 L_1 + \cdots + \mu_n L_n$.

The positive roots are $\{L_i - L_j\}_{i<j}$ and $\{L_i + L_j\}_{i \leq j}$, from which we find

$$\rho = \sum (n + 1 - i)L_i = L_1 + (L_1 + L_2) + \cdots + (L_1 + \cdots + L_n), \tag{24.12}$$

i.e., $\rho = (n, n-1, \ldots, 1)$.



As we saw in Lecture 16, an element in the Weyl group can be written uniquely as a product $\varepsilon\sigma$, where σ is a permutation of $\{L_1, \ldots, L_n\}$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_i = \pm 1$. Hence

$$A_\mu = \sum_\sigma (-1)^\sigma \sum_\varepsilon (-1)^\varepsilon e\left(\sum_{i=1}^{n} \varepsilon_i \mu_i L_{\sigma(i)}\right); \tag{24.13}$$

here the sign $(-1)^\varepsilon$ is the product of the ε_i. Now with $x_i = e(L_i)$, this can be written

$$A_\mu = \sum_\sigma (-1)^\sigma \prod_{i=1}^{n} (x_{\sigma(i)}^{\mu_i} - x_{\sigma(i)}^{-\mu_i})$$

or

$$A_\mu = |x_j^{\mu_i} - x_j^{-\mu_i}|, \tag{24.14}$$

where $|a_{i,j}|$ denotes the determinant of the $n \times n$ matrix $(a_{i,j})$. In particular,

$$A_\rho = |x_j^{n-i+1} - x_j^{-(n-i+1)}|. \tag{24.15}$$

From (24.14) or Exercise A.52 we have

$$A_\rho = \Delta(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}) \cdot (x_1 - x_1^{-1}) \cdot \ldots \cdot (x_n - x_n^{-1}), \tag{24.16}$$

where Δ is the discriminant.

Exercise 24.17. Show that

$$A_\rho = \prod_{i<j} (x_i - x_j)(x_i x_j - 1) \cdot \prod_i (x_i^2 - 1)/(x_1 \cdot \ldots \cdot x_n)^n.$$

The character of the irreducible representation Γ_λ with highest weight $\lambda = \sum \lambda_i L_i$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, is therefore:

$$\mathrm{Char}(\Gamma_\lambda) = \frac{|x_j^{\lambda_i+n-i+1} - x_j^{-(\lambda_i+n-i+1)}|}{|x_j^{n-i+1} - x_j^{-(n-i+1)}|}. \tag{24.18}$$

The dimension of Γ_λ is easily worked out from Corollary 24.6:

$$\begin{aligned}\dim(\Gamma_\lambda) &= \prod_{i<j} \frac{(l_i - l_j)}{(j-i)} \cdot \prod_{i\le j} \frac{(l_i + l_j)}{(2n+2-i-j)} \\ &= \prod_{i<j} \frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)} \cdot \prod_i \frac{l_i}{m_i},\end{aligned} \tag{24.19}$$

where $l_i = \lambda_i + n - i + 1$ and $m_i = n - i + 1$.

Exercise 24.20. Show that, setting $l_i' = \lambda_i + n - i$,

$$\dim(\Gamma_\lambda) = \frac{\prod_{i<j} (l_i' - l_j')(l_i' + l_j' + 2) \cdot \prod_i (l_i' + 1)}{(2n-1)! \cdot (2n-3)! \cdot \ldots \cdot 1!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\ldots,a_n}$ with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$, where the ω_i are the fundamental weights, using the relation $\lambda_i = a_i + \cdots + a_n$.

Exercise 24.21. Use Exercise 24.20 to verify that for $\lambda = L_1 + \cdots + L_k$, the dimension of Γ_λ is $2n$ if $k = 1$, and $\binom{2n}{k} - \binom{2n}{k-2}$ if $k \geq 2$. Use this to give another proof that the kernel of the contraction from $\bigwedge^k V$ to $\bigwedge^{k-2} V$ is irreducible.

The first determinantal formula for the symplectic group goes as follows. Let

$$J_d(x_1, \ldots, x_n) = H_d(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}),$$

where H_d is the dth complete symmetric polynomial in $2n$ variables. In other words, J_d is the character of the representation $\mathrm{Sym}^d(\mathbb{C}^{2n})$ of $\mathfrak{sp}_{2n}\mathbb{C}$. From Proposition A.50 of Appendix A we have

Proposition 24.22. *If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$, the character of Γ_λ is the determinant of the $r \times r$ matrix whose ith row is*

$$(J_{\lambda_i-i+1} \quad J_{\lambda_i-i+2} + J_{\lambda_i-i} \quad J_{\lambda_i-i+3} + J_{\lambda_i-i-1} \quad \cdots \quad J_{\lambda_i-i+r} + J_{\lambda_i-i-r+2}).$$

For example, for $\lambda = (d)$, i.e., $\lambda = dL_1$, we have $\mathrm{Char}(\Gamma_{(d)}) = J_d$, which is the character of $\mathrm{Sym}^d(\mathbb{C}^{2n})$. In particular, this verifies that the kth *symmetric powers* $\mathrm{Sym}^k(\mathbb{C}^{2n})$ *of the standard representation are all irreducible.* (This, of course, is a special case of the general description given in §17.3, since all the contraction maps vanish on the symmetric powers.)

Exercise 24.23. (i) Find the character of the representation of $\mathfrak{sp}_4\mathbb{C}$ with highest weight $\omega_1 + \omega_2 = 2L_1 + L_2$, verifying that the multiplicities are as we found in §16.2. (ii) Find the character of the representation of $\mathfrak{sp}_6\mathbb{C}$ with highest weight $\omega_1 + \omega_2$, thus verifying the assertion of Exercise 17.4.

The second Giambelli formula in the symplectic case expresses Γ_λ in terms of the basic representations

$$\Gamma_{\omega_k} = \mathrm{Ker}(\Lambda^k(\mathbb{C}^{2n}) \to \Lambda^{k-2}(\mathbb{C}^{2n}))$$

which are the kernels of the contractions. The character of Γ_{ω_k} is E'_k, where $E'_0 = 1$, $E'_1 = E_1 = x_1 + \cdots + x_n + x_1^{-1} + \cdots + x_n^{-1}$, and

$$E'_k = E_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) - E_{k-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$$

for $k \geq 2$, where E_k is the kth elementary symmetric polynomial. The formula is

Corollary 24.24. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose* ith *row is*

$$(E'_{\mu_i-i+1} \quad E'_{\mu_i-i+2} + E'_{\mu_i-i} \quad E'_{\mu_i-i+3} + E'_{\mu_i-i-1} \quad \cdots \quad E'_{\mu_i-i+l} + E'_{\mu_i-i-l+2}).$$

Proof. This follows from the proposition and Proposition A.44, which equates the two determinants before specializing the variables. □

There is also a simple formula for the character in terms of the characters E_k of $\Lambda^k(\mathbb{C}^{2n})$, which also follows from Proposition A.44:

$$\mathrm{Char}(\Gamma_\lambda) = |E_{\mu_i-i+j} - E_{\mu_i-i-j}|. \tag{24.25}$$

Note that $E_{n+k} = E_{n-k}$ (corresponding to the isomorphism $\Lambda^{n+k}\mathbb{C}^{2n} \cong \Lambda^{n-k}\mathbb{C}^{2n}$) and $E'_{n+k} = -E'_{n-k+2}$. In particular, Corollary 24.24 expresses $\mathrm{Char}(\Gamma_\lambda)$ as a polynomial in the characters of the basic representations $\Gamma_{\omega_1}, \ldots, \Gamma_{\omega_n}$.

The Odd Orthogonal Case

For $\mathfrak{so}_{2n+1}\mathbb{C}$ the weights are $\sum \mu_i L_i$, $\mu = (\mu_1, \ldots, \mu_n)$, with all μ_i integers or all half-integers. The positive roots are $\{L_i - L_j\}_{i<j}$, $\{L_i + L_j\}_{i<j}$, and $\{L_i\}$, so ρ is $\frac{1}{2}(L_1 + \cdots + L_n)$ less than in the case for $\mathfrak{sp}_{2n}$:

$$\rho = \sum (n + \tfrac{1}{2} - i)L_i, \tag{24.26}$$

or

$$\rho = (n - \tfrac{1}{2}, n - \tfrac{3}{2}, \ldots, \tfrac{1}{2}).$$

With $x_i^{\pm 1} = e(\pm L_i)$ and $x_i^{\pm 1/2} = e(\pm L_i/2)$, we have the same formula as before [(24.14)] for A_μ.

Exercise 24.27*. Show that

$$A_\rho = |x_j^{n-i+1/2} - x_j^{-(n-i+1/2)}|$$
$$= \Delta(x_1 + x_1^{-1}, \dots, x_n + x_n^{-1})\cdot(x_1^{1/2} - x_1^{-1/2})\cdot\ldots\cdot(x_n^{1/2} - x_n^{-1/2}).$$

If Γ_λ is the irreducible representation with highest weight $\lambda = \sum \lambda_i L_i$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, then the character formula can be written

$$\operatorname{Char}(\Gamma_\lambda) = \frac{|x_j^{\lambda_i+n-i+1/2} - x_j^{-(\lambda_i+n-i+1/2)}|}{|x_j^{n-i+1/2} - x_j^{-(n-i+1/2)}|}. \tag{24.28}$$

Similarly,

$$\dim(\Gamma_\lambda) = \prod_{i<j} \frac{(l_i - l_j)}{(j-i)} \cdot \prod_{i\leq j} \frac{(l_i + l_j)}{(2n+1-i-j)}$$
$$= \prod_{i<j} \frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)} \cdot \prod_i \frac{l_i}{m_i}, \tag{24.29}$$

where $l_i = \lambda_i + n - i + \frac{1}{2}$, and $m_i = n - i + \frac{1}{2}$.

Exercise 24.30. Show that, with $l_i' = \lambda_i + n - i$,

$$\dim(\Gamma_\lambda) = \frac{\prod_{i<j} (l_i' - l_j')(l_i' + l_j' + 1)\cdot \prod_i (2l_i' + 1)}{(2n-1)!\cdot(2n-3)!\cdot\ldots\cdot 1!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\dots,a_n}$ with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$, where the ω_i are the fundamental weights, using the equations

$$\lambda_i = a_i + \cdots + a_{n-1} + \tfrac{1}{2}a_n.$$

Exercise 24.31. Use the dimension formula to verify that for $\lambda = L_1 + \cdots + L_k$, the dimension of Γ_λ is $\binom{2n+1}{k}$. Use this to give another proof that $\bigwedge^k V$ is irreducible for $1 \leq k \leq n$. Verify that the dimension of the spin representation is 2^n, thus reproving that it is irreducible.

Exercise 24.32. Use the dimension formula to verify that the kernel of the contraction

$$\operatorname{Sym}^d(\mathbb{C}^{2n+1}) \to \operatorname{Sym}^{d-2}(\mathbb{C}^{2n+1})$$

is an irreducible representation with highest weight dL_1.

In case the representation is a representation of $\mathrm{SO}_{2n+1}\mathbb{C}$, i.e., the λ_i are all integral, there is a first determinantal formula that expresses Γ_λ in terms of the kernels of the contractions

$$\operatorname{Ker}(\operatorname{Sym}^d(\mathbb{C}^{2n+1}) \to \operatorname{Sym}^{d-2}(\mathbb{C}^{2n+1})).$$

Let K_d denote the character of this kernel, so $K_0 = 1$, $K_1 = x_1 + \cdots + x_n + x_1^{-1} + \cdots + x_n^{-1} + 1$, and

$$K_d = H_d(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1) - H_{d-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1),$$

where H_d is the dth complete symmetric polynomial. From Proposition A.60 we have

Proposition 24.33. *If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$, with the λ_i integral, then the character of Γ_λ is the determinant of the $r \times r$ matrix whose* ith *row is*

$$(K_{\lambda_i - i+1} \quad K_{\lambda_i - i+2} + K_{\lambda_i - i} \quad K_{\lambda_i - i+3} + K_{\lambda_i - i-1} \quad \cdots \quad K_{\lambda_i - i+r} + K_{\lambda_i - i-r+2}).$$

In particular, for $\lambda = (d)$, the character is K_d, which verifies that the kernel of $\mathrm{Sym}^d(\mathbb{C}^{2n+1}) \to \mathrm{Sym}^{d-2}(\mathbb{C}^{2n+1})$ is irreducible.

Exercise 24.34. Use the character formula to verify that the multiplicities of the representation $\Gamma_{2L_1+L_2}$ of $\mathfrak{so}_5\mathbb{C}$ are as specified in Exercise 18.9.

The second determinantal formula for $SO_{2n+1}\mathbb{C}$ writes Γ_λ in terms of the representations $\bigwedge^k(\mathbb{C}^{2n+1})$, whose characters are

$$E_k = E_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1).$$

Applying Proposition 24.33 with Corollary A.46, we have

Corollary 24.35. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose* ith *row is*

$$(E_{\mu_i - i+1} \quad E_{\mu_i - i+2} + E_{\mu_i - i} \quad \cdots \quad E_{\mu_i - i+l} + E_{\mu_i - i-l+2}).$$

Since $E_{n+k} = E_{n+1-k}$ (corresponding to the isomorphism $\bigwedge^{n+k}\mathbb{C}^{2n+1} \cong \bigwedge^{n+1-k}\mathbb{C}^{2n+1}$), this expresses $\mathrm{Char}(\Gamma_\lambda)$ as a polynomial in $E_1, \ldots, E_n$, with $E_d = \mathrm{Char}(\bigwedge^d\mathbb{C}^{2n+1})$.

The Even Orthogonal Case

For $\mathfrak{so}_{2n}\mathbb{C}$ the weights are the same as in the preceding case. This time the $\{L_i\}$ are not positive roots, however, so ρ is $\frac{1}{2}(L_1 + \cdots + L_n)$ less than in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$, or $L_1 + \cdots + L_n$ less than in the case of $\mathfrak{sp}_{2n}\mathbb{C}$:

$$\rho = \sum (n - i)L_i, \tag{24.36}$$

or

$$\rho = (n-1, n-2, \ldots, 0).$$

The calculation of A_μ is similar, but using only those ε of positive sign. This time

$$\sum_{\varepsilon}(-1)^{\varepsilon}e\left(\sum_{i=1}^{n}\varepsilon_i\mu_i L_{\sigma(i)}\right)=\frac{1}{2}\left[\prod_{i=1}^{n}(x_{\sigma(i)}^{\mu_i}+x_{\sigma(i)}^{-\mu_i})+\prod_{i=1}^{n}(x_{\sigma(i)}^{\mu_i}-x_{\sigma(i)}^{-\mu_i})\right].$$

This leads to

$$A_\mu=\tfrac{1}{2}(|x_j^{\mu_i}+x_j^{-\mu_i}|+|x_j^{\mu_i}-x_j^{-\mu_i}|). \tag{24.37}$$

Note that the second determinant term vanishes when any μ_i is zero. In particular,

$$A_\rho=\tfrac{1}{2}|x_j^{n-i}+x_j^{-(n-i)}|. \tag{24.38}$$

From (24.14) or Exercise A.66,

$$A_\rho=\Delta(x_1+x_1^{-1},\ldots,x_n+x_n^{-1}). \tag{24.39}$$

This gives, with Γ_λ the irreducible representation with highest weight $\lambda=\sum\lambda_i L_i$, $\lambda_1\geq\cdots\geq|\lambda_n|\geq 0$,

$$\operatorname{Char}(\Gamma_\lambda)=\frac{|x_j^{l_i}+x_j^{-l_i}|+|x_j^{l_i}-x_j^{-l_i}|}{|x_j^{n-i}+x_j^{-(n-i)}|}, \tag{24.40}$$

where $l_i=\lambda_i+n-i$. As before,

$$\begin{aligned}\dim(\Gamma_\lambda)&=\prod_{i<j}\frac{(l_i-l_j)}{(j-i)}\cdot\frac{(l_i+l_j)}{(2n-i-j)}\\&=\prod_{i<j}\frac{(l_i^2-l_j^2)}{(m_i^2-m_j^2)},\end{aligned} \tag{24.41}$$

where $l_i=\lambda_i+n-i$ and $m_i=n-i$. Note that, as expected, the two representations with weights $(\lambda_1,\ldots,\lambda_{n-1},\pm\lambda_n)$ have the same dimensions.

Exercise 24.42. Show that

$$\dim(\Gamma_\lambda)=2^{n-1}\frac{\prod_{i<j}(l_i-l_j)(l_i+l_j)}{(2n-2)!\cdot(2n-4)!\cdot\ldots\cdot 2!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\ldots,a_n}$ with highest weight $a_1\omega_1+\cdots+a_n\omega_n$, where the ω_i are the fundamental weights, using the equations

$$\lambda_i=a_i+\cdots+a_{n-2}+\tfrac{1}{2}(a_{n-1}+a_n),\quad 1\leq i\leq n-2,$$

$$\lambda_{n-1}=\tfrac{1}{2}(a_{n-1}+a_n),\qquad \lambda_n=\tfrac{1}{2}(-a_{n-1}+a_n).$$

Exercise 24.43. Use the dimension formula to verify that for $\omega=L_1+\cdots+L_k$, $k<n$, the dimension of Γ_ω is $\binom{2n}{k}$, so $\wedge^k(\mathbb{C}^{2n})$ is irreducible. For $\lambda=L_1+\cdots+L_{n-1}\pm L_n$, the dimension is $\frac{1}{2}\binom{2n}{k}$, so $\wedge^n(\mathbb{C}^{2n})$ is the sum of the two corresponding irreducible representations. Verify that the dimension of the two spin representations are 2^{n-1}, proving irreducibility again.

Note that the second term in the numerator in (24.40) changes sign when λ_n is replaced by $-\lambda_n$; in particular, it vanishes when $\lambda_n = 0$. When $\lambda_n = 0$, the representation Γ_λ is a representation of the orthogonal group $O_{2n}\mathbb{C}$. When $\lambda_n \neq 0$, the direct sum of the two representations with highest weights $(\lambda_1, \ldots, \pm\lambda_n)$ is an irreducible representation of $O_{2n}\mathbb{C}$. (See Exercises 23.19 and 23.37.)

Let L_d be the character of $\mathrm{Ker}(\mathrm{Sym}^d(\mathbb{C}^{2n}) \to \mathrm{Sym}^{d-2}(\mathbb{C}^{2n}))$, i.e., $L_d = H_d(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}) - H_{d-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n})$. In either case, Proposition A.64 applies to give the first determinantal formula:

Proposition 24.44. *Given integers $\lambda_1 \geq \cdots \geq \lambda_r > 0$, the character of the irreducible representation of $O_{2n}\mathbb{C}$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the determinant of the $r \times r$ matrix whose* i*th row is*

$$(L_{\lambda_i - i+1} \quad L_{\lambda_i - i+2} + L_{\lambda_i - i} \quad \cdots \quad L_{\lambda_i - i+r} + L_{\lambda_i - i - r+2}).$$

Again, for $\lambda = (d)$, this verifies that the kernel of the contraction from $\mathrm{Sym}^d(\mathbb{C}^{2n})$ to $\mathrm{Sym}^{d-2}(\mathbb{C}^{2n})$ is irreducible.

The second determinantal formula is the same as in the odd case, but with $E_k = E_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n})$:

Corollary 24.45. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose* i*th row is*

$$(E_{\mu_i - i+1} \quad E_{\mu_i - i+2} + E_{\mu_i - i} \quad \cdots \quad E_{\mu_i - i+l} + E_{\mu_i - i - l+2}).$$

Using the fact that $E_{n+k} = E_{n-k}$, this expresses $\mathrm{Char}(\Gamma_\lambda)$ as a polynomial in $E_1, \ldots, E_n$, with $E_d = \mathrm{Char}(\wedge^d \mathbb{C}^{2n})$.

Exercise 24.46*. For each of the orthogonal groups $O_m\mathbb{C}$, show that the character of the irreducible representation with highest weight λ can be written in the form

$$\mathrm{Char}(\Gamma_\lambda) = |h_{\lambda_i - i+j} - h_{\lambda_i - i - j}|,$$

where h_k is the character of $\mathrm{Sym}^k(\mathbb{C}^m)$. Another formula for the dimension of Γ_λ is obtained by substituting $\binom{m}{k}$ for h_k in this determinant.

There are other formulas expressing the characters of general representations in terms of simpler ones. Abramsky, Jahn, and King [A-J-K] give one that can be expressed by the *same* formula for the general linear, symplectic, and orthogonal groups. The general irreducible representations are given by partitions λ or Young diagrams, and in their formula the simpler representations are those corresponding to hooks. To express it, let $(a * b)$ denote the hook with horizontal leg of length $a+1$ and vertical leg of length $b+1$, i.e., the partition $(a+1, 1, \ldots, 1)$, with b 1's. More generally, given $\mathbf{a} = (a_1 > \cdots > a_r \geq 0)$ and $\mathbf{b} = (b_1 > \cdots > b_r \geq 0)$ with a_r or b_r nonzero, let $(\mathbf{a} * \mathbf{b})$ denote the partition whose Young diagram has legs of these lengths to the right of and below the r diagonal boxes (cf. Frobenius's notation, Exercise 4.17). Let $\chi_{(\mathbf{a}*\mathbf{b})}$ denote the character of the corresponding irreducible representation. Their formula is

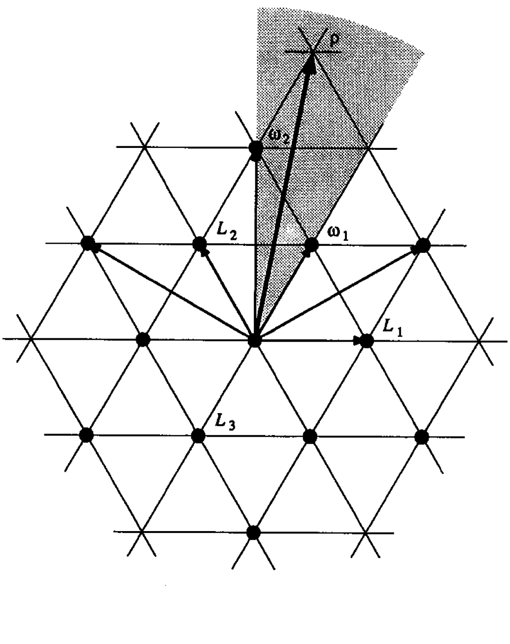
$$\chi_{(\mathbf{a}*\mathbf{b})} = |\chi_{(a_i*b_j)}|_{1\le i,j\le r}. \tag{24.47}$$

Taking the degree of both sides gives new formulas for the dimensions of the irreducible representations. These formulas are particularly useful if the rank r of the partition is small.

Exceptional Cases

We will, as a last example, work out the Weyl character formula for the exceptional Lie algebra $\mathfrak{g}_2$, and thereby verify some of the analysis of its representations given in Lecture 22. The remaining four exceptional Lie algebras we will leave as exercises.

To begin with, the value of ρ is easily seen to be $2L_1 + 3L_2$, in terms of the basis L_1, L_2 for the weight lattice introduced in Lecture 22.



Now, for any weight $\mu = pL_1 + qL_2 + rL_3$, we have

$$\begin{aligned} A_\mu &= \sum_{\sigma\in\mathfrak{S}_3} x_{\sigma(1)}^p\cdot x_{\sigma(2)}^q\cdot x_{\sigma(3)}^r - \sum_{\sigma\in\mathfrak{S}_3} x_{\sigma(1)}^{-p}\cdot x_{\sigma(2)}^{-q}\cdot x_{\sigma(3)}^{-r} \\ &= \Delta(x)\cdot S_{p,q,r}(x) - \Delta(x^{-1})\cdot S_{p,q,r}(x^{-1}), \end{aligned}$$

where we write x for (x_1, x_2, x_3) and x^{-1} for $(x_1^{-1}, x_2^{-1}, x_3^{-1})$, Δ is the discriminant, and $S_{p,q,r}$ the Schur function. Using the relation $\prod x_i = 1$ we can also write this as

$$= \Delta(x)\cdot(S_{p,q,r}(x) - S_{m-p,m-q,m-r}(x))$$

for any $m \geq \max(p, q, r)$. To make this notation agree with the standard notation for Schur polynomials from Appendix (A.4), note that $S_{p,q,r}$ is the Schur polynomial $S_{(s,t)}$ for the partition (s, t), $s \geq t$, where s is two less than the difference between the largest and smallest of p, q, and r, while t is one less than the difference between the second largest and the smallest; if p, q, and r are not distinct, $S_{p,q,r} = 0$. Thus, for example,

$$\begin{aligned} A_\rho &= \Delta(x)\cdot(S_{(1,1)}(x) - S_{(1)}(x)) \\ &= \Delta(x)\cdot(x_1x_2 + x_1x_3 + x_2x_3 - x_1 - x_2 - x_3). \end{aligned}$$

Now, any irreducible representation Γ_λ of $\mathfrak{g}_2$ has highest weight $\lambda = a\omega_1 + b\omega_2$, where $\omega_1 = L_1 + L_2$ and $\omega_2 = L_1 + 2L_2$ are the two fundamental weights, and a and b are non-negative integers. Then $\lambda + \rho = (a + b + 2)L_1 + (a + 2b + 3)L_2$. The Weyl character formula in this case becomes

Proposition 24.48. *The character of the representation of* $\mathfrak{g}_2$ *with highest weight* $a\omega_1 + b\omega_2$ *is*

$$\operatorname{Char}(\Gamma_{a,b}) = \frac{S_{(a+2b+1,a+b+1)} - S_{(a+2b+1,b)}}{S_{(1,1)} - S_{(1)}}.$$

Exercise 24.49. In the case of the standard representation $\Gamma_{1,0}$, the adjoint representation $\Gamma_{0,1}$, and the representation $\Gamma_{2,0}$, use this formula to verify the multiplicities found in Lecture 22.

We can also work out the dimension formula explicitly in this case. The two fundamental weights ω_1 and ω_2 have inner products

$$(\omega_1, \omega_1) = 1, \qquad (\omega_1, \omega_2) = 3/2, \qquad \text{and} \qquad (\omega_2, \omega_2) = 3;$$

ω_1 and ω_2 are among the positive roots of $\mathfrak{g}_2$, and in terms of these the remaining positive roots are $2\omega_1 - \omega_2$, $3\omega_1 - \omega_2$, $\omega_2 - \omega_1$, and $2\omega_2 - 3\omega_1$. The weight ρ is the sum of the fundamental weights ω_1 and ω_2, so that for an arbitrary weight $\lambda = a\omega_1 + b\omega_2$ we have the following table of inner products:

	$(\cdot, \rho)$	$(\cdot, \lambda)$	$(\cdot, \lambda + \rho)$
$2\omega_1 - \omega_2$	$1/2$	$a/2$	$(a + 1)/2$
$3\omega_1 - \omega_2$	3	$3a/2 + 3b/2$	$3(a + b + 2)/2$
ω_1	$5/2$	$a + 3b/2$	$(2a + 3b + 5)/2$
ω_2	$9/2$	$3a/2 + 3b$	$3(a + 2b + 3)/2$
$-\omega_1 + \omega_2$	2	$a/2 + 3b/2$	$(a + 3b + 4)/2$
$-3\omega_1 + 2\omega_2$	$3/2$	$3b/2$	$3(b + 1)/2$

We conclude that *the dimension of the irreducible representation* $\Gamma_{a,b}$ *of* $\mathfrak{g}_2$ *with highest weight* $\lambda = a\omega_1 + b\omega_2$ *is*

$$\dim(\Gamma_{a,b}) = \frac{(a+1)(a+b+2)(2a+3b+5)(a+2b+3)(a+3b+4)(b+1)}{120}.$$

We can check this in the cases $a = 1$, $b = 0$ and $a = 0$, $b = 1$, getting the dimensions 7 and 14 of the standard and adjoint representations, respectively. In case $a = 2$, $b = 0$ we may verify the result of the explicit calculation in Lecture 22, finding that

$$\dim(\Gamma_{2,0}) = 27$$

and, therefore, deducing that $\wedge^3 V = \Gamma_{2,0} \oplus V \oplus \mathbb{C}$ and $\mathrm{Sym}^2 V = \Gamma_{2,0} \oplus \mathbb{C}$.

Exercise 24.50. Show that $\mathrm{Sym}^a V = \bigoplus_{k=0}^{[a/2]} \Gamma_{a-2k,0}$.

We leave the analogous computations for the remaining four Lie algebras as exercises, using the description of the root systems found in Exercise 21.16. Since we have not said much about the Weyl group in the exceptional cases the formula (WCF) cannot be used directly—not to mention the fact that the orders of these Weyl groups are: $2^7 \cdot 3^2 = 1152$ for $\mathfrak{f}_4$; $2^7 \cdot 3^4 \cdot 5 = 51{,}840$ for $\mathfrak{e}_6$, $2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2{,}903{,}040$ for $\mathfrak{e}_7$, and $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696{,}729{,}600$ for $\mathfrak{e}_8$. However, the dimension formula is available.

Exercise 24.51*. For each of the four remaining exceptional Lie algebras, compute ρ = half the sum of the positive roots. For each of the fundamental weights ω, at least for $\mathfrak{f}_4$, compute the dimension of the irreducible representation with highest weight ω. In particular, find the nontrivial representation of minimal dimension. Use this to verify that (E_6) is not isomorphic to (B_6) or (C_6), i.e., that $\mathfrak{e}_6$ is not isomorphic to $\mathfrak{so}_{13}\mathbb{C}$ or $\mathfrak{sp}_{12}\mathbb{C}$.

Exercise 24.52*. List all irreducible representations V of simple Lie algebras $\mathfrak{g}$ such that $\dim V \leq \dim \mathfrak{g}$. Note that these include all cases where the corresponding group representation has a Zariski dense orbit, or a finite number of orbits.

LECTURE 25

More Character Formulas

In this lecture we give two more formulas for the multiplicities of an irreducible representation of a semisimple Lie algebra or group. First, Freudenthal's formula (§25.1) gives a straightforward way of calculating the multiplicity of a given weight once we know the multiplicity of all higher ones. This in turn allows us to prove in §25.2 the Weyl character formula, as well as another multiplicity formula due to Kostant. Finally, in §25.3 we give Steinberg's formula for the decomposition of the tensor product of two arbitrary irreducible representations of a semisimple Lie algebra, and also give formulas for some pairs $\mathfrak{h} \subset \mathfrak{g}$ for the decomposition of the restriction to $\mathfrak{h}$ of irreducible representations of $\mathfrak{g}$.

§25.1. Freudenthal's Multiplicity Formula

Freudenthal's formula gives a general way of computing the multiplicities of a representation, i.e., the dimensions of its weight spaces, by working down successively from the highest weight. The result is similar to (but more complicated than) what we did for $\mathfrak{sl}_3\mathbb{C}$ in Lecture 13, where we found the multiplicities along successive concentric hexagons in the weight diagram.

Let Γ_λ be the irreducible representation with highest weight λ, which will be fixed throughout this discussion. Let $n_\mu = n_\mu(\Gamma_\lambda)$ be the dimension of the weight space[1] of weight μ in Γ_λ, i.e., $\mathrm{Char}(\Gamma_\lambda) = \sum n_\mu e(\mu)$. Freudenthal gives a formula for n_μ in terms of multiplicities of weights that are higher than μ.

[1] In the literature, these multiplicities n_μ are often referred to as "inner multiplicities."

Proposition 25.1 (Freudenthal's Multiplicity Formula). *With the above notation,*

$$c(\mu)\cdot n_\mu(\Gamma_\lambda) = 2\sum_{\alpha\in R^+}\sum_{k\geq 1}(\mu + k\alpha, \alpha)n_{\mu+k\alpha},$$

where $c(\mu) = \|\lambda + \rho\|^2 - \|\mu + \rho\|^2$.

Here $\|\beta\|^2 = (\beta, \beta)$, $(\ ,\ )$ is the Killing form, and ρ is half the sum of the positive roots.

Exercise 25.2*. Verify that $c(\mu)$ is positive if $\mu \neq \lambda$ and $n_\mu > 0$.

The proof of Freudenthal's formula uses a *Casimir operator*, denoted C. This is an endomorphism of any representation V of the semisimple Lie algebra $\mathfrak{g}$, and is constructed as follows. Take any basis $U_1, \ldots, U_r$ for $\mathfrak{g}$, and let $U_1', \ldots, U_r'$ be the dual basis with respect to the Killing form on $\mathfrak{g}$. Set

$$C = U_1U_1' + \cdots + U_rU_r',$$

i.e., for any $v \in V$, $C(v) = \sum U_i\cdot(U_i'\cdot v)$.

Exercise 25.3. Verify that C is independent of the choice of basis[2].

The key fact is

Exercise 25.4*. Show that C commutes with every operation in $\mathfrak{g}$, i.e.,

$$C(X\cdot v) = X\cdot C(v) \quad \text{for all } X \in \mathfrak{g},\ v \in V.$$

The idea is to use a special basis for the construction of C, so that each term U_iU_i' will act as multiplication by a constant on any weight space, and this constant can be calculated in terms of multiplicities. Then Schur's lemma can be applied to know that, in case V is irreducible, C itself is multiplication by a scalar. Taking traces will lead to a relation among multiplicities, and a little algebraic manipulation will give Freudenthal's formula.

The basis for $\mathfrak{g}$ to use is a natural one: Choose the basis $H_1, \ldots, H_n$ for the Cartan subalgebra $\mathfrak{h}$, where $H_i = H_{\alpha_i}$ corresponds to the simple root α_i, and let H_i' be the dual basis for the restriction of the Killing form to $\mathfrak{h}$. For each root α, choose a nonzero $X_\alpha \in \mathfrak{g}_\alpha$. The dual basis will then have X_α' in $\mathfrak{g}_{-\alpha}$. In fact, if we let $Y_\alpha \in \mathfrak{g}_{-\alpha}$ be the usual element so that X_α, Y_α, and $H_\alpha = [X_\alpha, Y_\alpha]$ are the canonical basis for the subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$ that they span, then

$$X_\alpha' = ((\alpha, \alpha)/2)\,Y_\alpha. \tag{25.5}$$

Exercise 25.6*. Verify (25.5) by showing that $(X_\alpha, Y_\alpha) = 2/(\alpha, \alpha)$.

[2] In fancy language, C is an element of the universal enveloping algebra of $\mathfrak{g}$, but we do not need this.

Now we have the Casimir operator

$$C = \sum H_i H_i' + \sum_{\alpha \in R} X_\alpha X_\alpha',$$

and we analyze the action of C on the weight space V_μ corresponding to weight μ for any representation V. Let $n_\mu = \dim(V_\mu)$. First we have

$$\sum H_i H_i' \text{ acts on } V_\mu \text{ by multiplication by } (\mu, \mu) = \|\mu\|^2. \tag{25.7}$$

Indeed, $H_i H_i'$ acts by multiplication by $\mu(H_i)\mu(H_i')$. If we write $\mu = \sum r_i \omega_i$, where the ω_i are the fundamental weights, then $\mu(H_i) = r_i$, and if $\mu = \sum r_i' \omega_i'$, with ω_i' the dual basis to ω_i, then similarly $\mu(H_i') = r_i'$. Hence $\sum \mu(H_i)\mu(H_i') = \sum r_i r_i' = (\mu, \mu)$, as asserted.

Now consider the action of $X_\alpha X_\alpha' = ((\alpha, \alpha)/2) X_\alpha Y_\alpha$ on V_μ. Restricting to the subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ and to the subrepresentation $\bigoplus_i V_{\mu + i\alpha}$ corresponding to the α-string through μ, we are in a situation which we know very well. Suppose this string is

$$V_\beta \oplus V_{\beta - \alpha} \oplus \cdots \oplus V_{\beta - m\alpha},$$

so $m = \beta(H_\alpha)$ [cf. (14.10)], and let k be the integer such that $\mu = \beta - k\alpha$. We assume for now that $k \le m/2$.

On the first term V_β, $X_\alpha Y_\alpha$ acts by multiplication by $m = \beta(H_\alpha) = 2(\beta, \alpha)/(\alpha, \alpha)$, so $X_\alpha X_\alpha'$ acts by multiplication by (β, α). In general, on the part of $V_{\beta - k\alpha}$ which is the image of V_β by multiplication by $(Y_\alpha)^k$, we know [cf. (11.5)] that $X_\alpha Y_\alpha$ acts by multiplication by $(k + 1)(m - k)$. This gives us a subspace of V_μ of dimension n_β on which $X_\alpha X_\alpha'$ acts by multiplication by

$$(k + 1)((\beta, \alpha) - k(\alpha, \alpha)/2) = (k + 1)((\mu, \alpha) + k(\alpha, \alpha)/2).$$

Now peel off the subrepresentation (over $\mathfrak{s}_\alpha$) of V spanned by V_β, and apply the same reasoning to what is left. We have a subspace of $V_{\beta - \alpha}$ of dimension $n_{\beta - \alpha} - n_\beta$ to which the same analysis can be made. From this we get a subspace of V_μ of dimension $n_{\beta - \alpha} - n_\beta$ on which $X_\alpha X_\alpha'$ acts by multiplication by

$$(k)((\mu, \alpha) + (k - 1)(\alpha, \alpha)/2).$$

Continuing to peel off subrepresentations, the space V_μ is decomposed into pieces on which $X_\alpha X_\alpha'$ acts by multiplication by a scalar. The trace of $X_\alpha X_\alpha'$ on V_μ is therefore the sum

$$\begin{aligned} &n_\beta \cdot (k + 1)((\mu, \alpha) + k(\alpha, \alpha)/2) + (n_{\beta - \alpha} - n_\beta) \cdot (k)((\mu, \alpha) + (k - 1)(\alpha, \alpha)/2) \\ &\quad + \cdots + ((n_{\beta - k\alpha} - n_{\beta - (k-1)\alpha}) \cdot (1)((\mu, \alpha) + (0)(\alpha, \alpha)/2). \end{aligned}$$

Canceling in successive terms, this simplifies to

$$\mathrm{Trace}(X_\alpha X_\alpha'|_{V_\mu}) = \sum_{i=0}^{k} (\mu + i\alpha, \alpha) n_{\mu + i\alpha}. \tag{25.8}$$

One pleasant fact about this sum is that it may be extended to all $i \ge 0$, since $n_{\mu + i\alpha} = 0$ for $i > k$.

In case $k \geq m/2$, the computation is similar, peeling off representations from the other end, starting with $V_{\beta-m\alpha}$. The only difference is that the action of $X_\alpha Y_\alpha$ on $V_{\beta-m\alpha}$ is zero. The result is

$$\operatorname{Trace}(X_\alpha X'_\alpha|_{V_\mu}) = -\sum_{i=1}^{\infty} (\mu - i\alpha, \alpha) n_{\mu-i\alpha}. \tag{25.9}$$

Exercise 25.10. Show that $X_\alpha X'_\alpha = X_{-\alpha} X'_{-\alpha} + ((\alpha, \alpha)/2) H_\alpha$, and deduce (25.9) directly from (25.8) by replacing α by $-\alpha$.

In fact, (25.8) is valid for all μ and α, as we see from the identity

$$\sum_{i=-\infty}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} = 0. \tag{25.11}$$

Exercise 25.12*. Verify (25.11) by using the symmetry of the α-string through β.

Now we add the assumption that V is irreducible, so C is multiplication by some scalar c. Taking the trace of C on V_μ and adding, we get

$$c n_\mu = (\mu, \mu) n_\mu + \sum_{\alpha \in R} \sum_{i \geq 0} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}. \tag{25.13}$$

Note that when $i = 0$ the two terms for α and $-\alpha$ cancel each other, so the summation can begin at $i = 1$ instead. Rewriting this in terms of the positive weights, and using (25.11) the sums become

$$\sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} + \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu - i\alpha, \alpha) n_{\mu-i\alpha}$$

$$= n_\mu \sum_{\alpha \in R^+} (\mu, \alpha) + 2 \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}.$$

Summarizing, and observing that $\sum_{\alpha \in R^+} (\mu, \alpha) = (\mu, 2\rho)$, we have

$$c n_\mu = ((\mu, \mu) + (\mu, 2\rho)) n_\mu + 2 \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}.$$

Note that $(\mu, \mu) + (\mu, 2\rho) = (\mu + \rho, \mu + \rho) - (\rho, \rho) = \|\mu + \rho\|^2 - \|\rho\|^2$. To evaluate the constant we evaluate on the highest weight space V_λ, where $n_\lambda = 1$ and $n_{\lambda+i\alpha} = 0$ for $i > 0$. Hence,

$$c = (\lambda, \lambda) + (\lambda, 2\rho) = \|\lambda + \rho\|^2 - \|\rho\|^2. \tag{25.14}$$

Combining the preceding two equations yields Freudenthal's formula. □

Exercise 25.15. Apply Freudenthal's formula to the representations of $\mathfrak{sl}_3\mathbb{C}$ considered in §13.2, verifying again that the multiplicities are as prescribed on the hexagons and triangles.

Exercise 25.16. Use Freudenthal's formula to calculate multiplicities for the representations $\Gamma_{1,0}$, $\Gamma_{0,1}$, and $\Gamma_{2,0}$ of $(\mathfrak{g}_2)$.

§25.2. Proof of (WCF); the Kostant Multiplicity Formula

It is not unreasonable to anticipate that Weyl's character formula can be deduced from Freudenthal's inductive formula, but some algebraic manipulation is certainly required. Let

$$\chi_\lambda = \operatorname{Char}(\Gamma_\lambda) = \sum n_\mu e(\mu)$$

be the character of the irreducible representation with highest weight λ. Freudenthal's formula, in form (25.13), reads[3]

$$c \cdot \chi_\lambda = \sum_\mu (\mu, \mu) n_\mu e(\mu) + \sum_\mu \sum_{\alpha \in R} \sum_{i=0}^{\infty} (\mu + i\alpha, \alpha) n_{\mu + i\alpha} e(\mu),$$

where $c = \|\lambda + \rho\|^2 - \|\rho\|^2$. To get this to look anything like Weyl's formula, we must get rid of the inside sums over i. If α is fixed, they will disappear if we multiply by $e(\alpha) - 1$, as successive terms cancel:

$$(e(\alpha) - 1) \cdot \sum_\mu \sum_{i=0}^{\infty} (\mu + i\alpha, \alpha) n_{\mu + i\alpha} e(\mu) = \sum_\mu (\mu, \alpha) n_\mu e(\mu + \alpha).$$

Let $P = \prod_{\alpha \in R} (e(\alpha) - 1) = (e(\alpha) - 1) \cdot P_\alpha$, where $P_\alpha = \prod_{\beta \neq \alpha} (e(\beta) - 1)$. The preceding two formulas give

$$c \cdot P \cdot \chi_\lambda = P \cdot \sum_\mu (\mu, \mu) n_\mu e(\mu) + \sum_{\mu, \alpha} (\mu, \alpha) P_\alpha n_\mu e(\mu + \alpha). \tag{25.17}$$

Note also that

$$P = (-1)^r A_\rho \cdot A_\rho,$$

where r is the number of positive roots, so at least the formula now involves the ingredients that go into (WCF).

We want to prove (WCF): $A_\rho \cdot \chi_\lambda = A_{\lambda + \rho}$. We have seen in §24.1 that both sides of this equation are alternating, and that both have highest weight term $e(\lambda + \rho)$, with coefficient 1. On the right-hand side the only terms that appear are those of the form $\pm e(W(\lambda + \rho))$, for W in the Weyl group. To prove (WCF), it suffices to prove that the only terms appearing with nonzero coefficients in $A_\rho \cdot \chi_\lambda$ are these same $e(W(\lambda + \rho))$, for then the alternating property and the knowledge of the coefficient of $e(\lambda + \rho)$ determine all the coefficients. This can be expressed as:

[3] In this section we work in the ring $\mathbb{C}[\Lambda]$ of finite sums $\sum m_\mu e(\mu)$ with complex coefficients m_μ.

Claim. *The only terms $e(\nu)$ occurring in $A_\rho \cdot \chi_\lambda$ with nonzero coefficient are those with $\|\nu\| = \|\lambda + \rho\|$.*

To see that this is equivalent, note that by definition of A_ρ and χ_λ, the terms in $A_\rho \cdot \chi_\lambda$ are all of the form $\pm e(\nu)$, where $\nu = \mu + W(\rho)$, for μ a weight of Γ_λ and W in the Weyl group. But if $\|\mu + W(\rho)\| = \|\lambda + \rho\|$, since the metric is invariant by the Weyl group, this gives $\|W^{-1}(\mu) + \rho\| = \|\lambda + \rho\|$. But we saw in Exercise 25.2 that this cannot happen unless $\mu = W(\lambda)$, as required.

We are thus reduced to proving the claim. This suggests looking at the "Laplacian" operator that maps $e(\mu)$ to $\|\mu\|^2 e(\mu)$, that is, the map

$$\Delta\colon \mathbb{C}[\Lambda] \to \mathbb{C}[\Lambda]$$

defined by

$$\Delta(\textstyle\sum m_\mu e(\mu)) = \sum (\mu, \mu) m_\mu e(\mu).$$

The claim is equivalent to the assertion that $F = A_\rho \cdot \chi_\lambda$ satisfies the "differential equation"

$$\Delta(F) = \|\lambda + \rho\|^2 F.$$

From the definition $\Delta(\chi_\lambda) = \sum (\mu, \mu) n_\mu e(\mu)$. And $\Delta(A_\rho) = \|\rho\|^2 A_\rho$. In general, since $\|W(\alpha)\| = \|\alpha\|$ for all $W \in \mathfrak{W}$,

$$\Delta(A_\alpha) = \textstyle\sum (-1)^W \|W(\alpha)\|^2 e(W(\alpha)) = \|\alpha\|^2 A_\alpha.$$

So we would be in good shape if we had a formula for Δ of a product of two functions. One expects such a formula to take the form

$$\Delta(fg) = \Delta(f)g + 2(\nabla f, \nabla g) + f\Delta(g), \tag{25.18}$$

where ∇ is a "gradient," and (,) is an "inner product." Taking $f = e(\mu)$, $g = e(\nu)$, we see that we need to have $(\nabla e(\mu), \nabla e(\nu)) = (\mu, \nu) e(\mu + \nu)$. There is indeed such a gradient and inner product. Define a homomorphism

$$\nabla\colon \mathbb{C}[\Lambda] \to \mathfrak{h}^* \otimes \mathbb{C}[\Lambda] = \operatorname{Hom}(\mathfrak{h}, \mathbb{C}[\Lambda])$$

by the formula $\nabla(e(\mu)) = \mu \cdot e(\mu)$, and define the bilinear form (,) on $\mathfrak{h}^* \otimes \mathbb{C}[\Lambda]$ by the formula $(\alpha e(\mu), \beta e(\nu)) = (\alpha, \beta) e(\mu + \nu)$, where (α, β) is the Killing form on $\mathfrak{h}^*$.

Exercise 25.19. With these definitions, verify that (25.18) is satisfied, as well as the Leibnitz rule

$$\nabla(fg) = \nabla(f)g + f\nabla(g).$$

For example, $\nabla(\chi_\lambda) = \sum_\mu n_\mu \mu \cdot e(\mu)$, and, by the Leibnitz rule,

$$\nabla(P) = \sum_{\alpha \in R} P_\alpha \alpha \cdot e(\alpha).$$

But now look at formula (25.17). This reads

$$c \cdot P\chi_\lambda = P\Delta(\chi_\lambda) + (\nabla P, \nabla \chi_\lambda).$$

Since, also by the exercise, $\nabla(P) = 2(-1)^r A_\rho \nabla(A_\rho)$, we may cancel $(-1)^r A_\rho$ from each term in the equation, getting

$$c \cdot A_\rho \chi_\lambda = A_\rho \Delta(\chi_\lambda) + 2(\nabla A_\rho, \nabla \chi_\lambda).$$

By the identity (25.18), the right-hand side of this equation is

$$\Delta(A_\rho \chi_\lambda) - \Delta(A_\rho)\chi_\lambda = \Delta(A_\rho \chi_\lambda) - \|\rho\|^2 A_\rho \chi_\lambda.$$

Since $c = \|\lambda + \rho\|^2 - \|\rho\|^2$, this gives $\|\lambda + \rho\|^2 A_\rho \chi_\lambda = \Delta(A_\rho \chi_\lambda)$, which finishes the proof. □

We conclude this section with a proof of another general multiplicity formula, discovered by Kostant. It gives an elegant closed formula for the multiplicities, but at the expense of summing over the entire Weyl group (although as we will indicate below, there are many interesting cases where all but a few terms of the sum vanish). It also involves a kind of partition counting function. For each weight μ, let $P(\mu)$ be the number of ways to write μ as a sum of positive roots; set $P(0) = 1$. Equivalently,

$$\prod_{\alpha \in R^+} \frac{1}{1 - e(\alpha)} = \sum_\mu P(\mu) e(\mu). \tag{25.20}$$

Proposition 25.21. (Kostant's Multiplicity Formula). *The multiplicity $n_\mu(\Gamma_\lambda)$ of weight μ in the irreducible representation Γ_λ is given by*

$$n_\mu(\Gamma_\lambda) = \sum_{W \in \mathfrak{W}} (-1)^W P(W(\lambda + \rho) - (\mu + \rho)),$$

where ρ is half the sum of the positive roots.

Proof. Write $(A_\rho)^{-1} = e(-\rho)/\prod(1 - e(-\alpha)) = \sum_\nu P(\nu) e(-\nu - \rho)$. By (WCF),

$$\begin{aligned} \chi_\lambda = A_{\lambda+\rho}(A_\rho)^{-1} &= \sum_{W,\nu} (-1)^W e(W(\lambda + \rho)) P(\nu) e(-\nu - \rho) \\ &= \sum_{W,\nu} (-1)^W P(\nu) e(W(\lambda + \rho) - (\nu + \rho)) \\ &= \sum_{W,\mu} (-1)^W P(W(\lambda + \rho) - (\mu + \rho)) e(\mu), \end{aligned}$$

as seen by writing $\mu = W(\lambda + \rho) - (\nu + \rho)$. □

In fact, the proof shows that Kostant's formula is equivalent to Weyl's formula, cf. [Cart].

One way to interpret Kostant's formula, at least for weights μ close to the highest weight λ of Γ_λ, is as a sort of converse to Proposition 14.13(ii). Recall that this says that Γ_λ will be generated by the images of its highest weight vector v under successive applications of the generators of the negative root spaces; in practice, we used this fact to bound from above the multiplicities of various weights μ close to λ by counting the number of ways of getting from λ to μ by adding negative roots. The problem in making this precise was always that we did not know how many relations there were among these images, if any. Kostant's formula gives an answer: for example, if the difference $\lambda - \mu$ is small relative to λ, we see that the only nonzero term in the sum is the principle term, corresponding to $W = 1$; in this case the answer is that there are no relations other than the trivial ones $X(Y(v)) - Y(X(v)) = [X, Y](v)$. When μ gets somewhat smaller, other terms appear corresponding to single reflections W in the walls of the Weyl chamber for which $W(\lambda + \rho)$ is higher than $\mu + \rho$; we can think of these terms, which all appear with sign -1, as correction terms indicating the presence of relations. As μ gets smaller still, of course, more terms appear of both signs, and this viewpoint breaks down.

To see how this works in practice, the reader can for example carry out the analysis of the example at the end of §13.1.

Exercise 25.22* (Kostant). Prove the following formula for the function P, which can be used to calculate it inductively: $P(0) = 1$, and, for $\mu \neq 0$,

$$P(\mu) = -\sum_{W \neq 1} (-1)^W P(\mu + W(\rho) - \rho).$$

Exercise 25.23* (Racah). Deduce from Kostant's formula and the preceding exercise the following inductive formula for the multiplicities n_μ of μ in Γ_λ: $n_\mu = 1$ if $\mu = \lambda$, and if μ is any other weight of Γ_λ, then

$$n_\mu = -\sum_{W \neq 1} (-1)^W n_{\mu+\rho-W(\rho)}.$$

Show, in fact, that for any weight μ

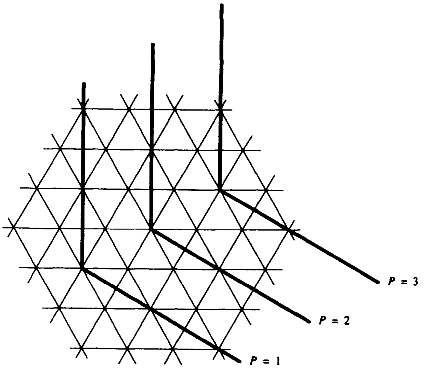
$$\sum_{W \in \mathfrak{W}} (-1)^W n_{\mu+\rho-W(\rho)} = \sum_{W'} (-1)^{W'},$$

where the second sum is over those $W' \in \mathfrak{W}$ such that $W'(\lambda + \rho) = \mu + \rho$.

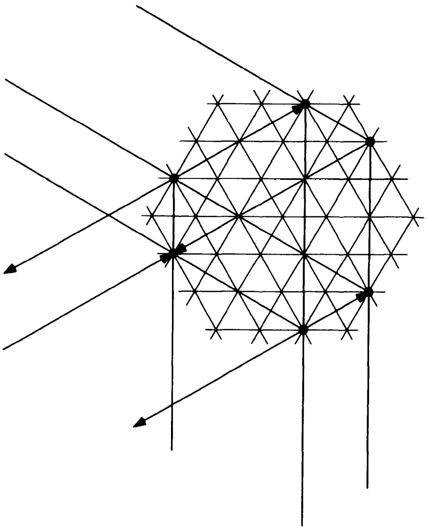
Note that Kostant's formula, more than any of the others, shows us directly the pattern of multiplicities in the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$. For one thing, it is easy to represent the function P diagrammatically: in the weight lattice of $\mathfrak{sl}_3\mathbb{C}$, the function $P(\mu)$ will be a constant 1 on the rays $\{aL_2 - aL_1\}_{a\geq 0}$ and $\{aL_3 - aL_2\}_{a\geq 0}$ through the origin in the direction of the two simple positive roots $L_2 - L_1$ and $L_3 - L_2$. It will have value 2 on the translates $\{aL_2 - (a+3)L_1\}_{a\geq -1}$ and $\{aL_3 - (a-3)L_2\}_{a\geq 2}$ of these two rays by the third positive root $L_3 - L_1$: for example, the first of these can be written as

$$\begin{aligned} aL_2 - (a+3)L_1 &= (a+1)\cdot(L_2 - L_1) + L_3 - L_1 \\ &= (a+2)\cdot(L_2 - L_1) + L_3 - L_2; \end{aligned}$$

and correspondingly its value will increase by 1 on each successive translate of these rays by $L_3 - L_1$. The picture is thus



Now, the prescription given in the Kostant formula for the multiplicities is to take six copies of this function flipped about the origin, translated so that the vertex of the outer shell lies at the points $w(\lambda + \rho) - \rho$ and take their alternating sum. Superimposing the six pictures we arrive at



which shows us clearly the hexagonal pattern of the multiplicities.

Exercise 25.24*. A nonzero dominant weight λ of a simple Lie algebra is called *minuscule* if $\lambda(H_\alpha) = 0$ or 1 for each positive root α.

(a) Show that if λ is minuscule, then every weight space of Γ_λ is one dimensional.

(b) Show that λ is minuscule if and only if all the weights of Γ_λ are conjugate under the Weyl group.
(c) Show that a minuscule weight must be one of the fundamental weights. Find the minuscule weights for each simple Lie algebra.

§25.3. Tensor Products and Restrictions To Subgroups

In the case of the general or special linear groups, we saw general formulas for describing how the tensor product $\Gamma_\lambda \otimes \Gamma_\mu$ of two irreducible representations decomposes:

$$\Gamma_\lambda \otimes \Gamma_\mu = \bigoplus_\nu N_{\lambda\mu\nu}\Gamma_\nu.$$

In these cases the multiplicities $N_{\lambda\mu\nu}$ can be described by a combinatorial formula: the Littlewood–Richardson rule. In general, such a decomposition is equivalent to writing

$$\chi_\lambda \chi_\mu = \sum_\nu N_{\lambda\mu\nu}\chi_\nu \tag{25.25}$$

in $\mathbb{Z}[\Lambda]$, where $\chi_\lambda = \mathrm{Char}(\Gamma_\lambda)$ denotes the character.[4] By Weyl's character formula, these multiplicities $N_{\lambda\mu\nu}$ are determined by the identity

$$A_{\lambda+\rho}\cdot A_{\mu+\rho} = \sum_\nu N_{\lambda\mu\nu}A_\rho \cdot A_{\nu+\rho}. \tag{25.26}$$

This formula gives an effective procedure for calculating the coefficients $N_{\lambda\mu\nu}$, if one that is tedious in practice: we can peel off highest weights, i.e., successively subtract from $A_{\lambda+\rho}\cdot A_{\mu+\rho}$ multiples of $A_\rho \cdot A_{\nu+\rho}$ for the highest ν that appears.

There are some explicit formulas for the other classical groups. R. C. King [Ki2] has showed that for both the symplectic or orthogonal groups, the multiplicities $N_{\lambda\mu\nu}$ are given by the formula

$$N_{\lambda\mu\nu} = \sum_{\zeta,\sigma,\tau} M_{\zeta\sigma\lambda}\cdot M_{\zeta\tau\mu}\cdot M_{\sigma\tau\nu}, \tag{25.27}$$

where the M's denote the Littlewood–Richardson multiplicities, i.e., the corresponding numbers for the general linear group, and the sum is over all partitions ζ, σ, τ. For other formulas for the classical groups, see [Mur1], [Wel, p. 230].

Exercise 25.28*. For $\mathfrak{so}_4\mathbb{C}$, show that all the nonzero multiplicities $N_{\lambda\mu\nu}$ are 1's, and these occur for ν in a rectangle with sides making 45° angles to the axes. Describe this rectangle.

[4] In the literature these multiplicities $N_{\lambda\mu\nu}$ are often called "outer multiplicities," and the problem of finding them, or decomposing the tensor product, the "Clebsch–Gordan" problem.

Steinberg has also given a general formula for the multiplicities $N_{\lambda\mu\nu}$. Since it involves a double summation over the Weyl group, using it in a concrete situation may be a challenge.

Proposition 25.29 (Steinberg's Formula). *The multiplicity of* Γ_ν *in* $\Gamma_\lambda \otimes \Gamma_\mu$ *is*

$$N_{\lambda\mu\nu} = \sum_{W,W'} (-1)^{WW'} P(W(\lambda + \rho) + W'(\mu + \rho) - \nu - 2\rho),$$

where the sum is over pairs $W, W' \in \mathfrak{W}$, *and* P *is the counting function appearing in Kostant's multiplicity formula.*

Exercise 25.30*. Prove Steinberg's formula by multiplying (25.25) by A_ρ, using (WCF) to get $\chi_\lambda A_{\mu+\rho} = \sum N_{\lambda\mu\nu} A_{\nu+\rho}$. Write out both sides, using Kostant's formula for χ_λ, and compute the coefficient of the term $e(\beta + \rho)$ on each side, for any β. This gives

$$\sum_{W,W'} (-1)^{WW'} P(W(\lambda + \rho) + W'(\mu + \rho) - \beta - 2\rho) = \sum_{W} (-1)^{W} N_{\lambda,\mu,W(\beta+\rho)-\rho}.$$

Show that for $\beta = \nu$ all the terms on the right are zero but $N_{\lambda\mu\nu}$.

Exercise 25.31 (Racah). Use the Steinberg and Kostant formulas to show that

$$N_{\lambda\mu\nu} = \sum_{W} (-1)^{W} n_{\nu+\rho-W(\mu+\rho)}(\Gamma_\lambda).$$

The following is the generalization of something we have seen several times:

Exercise 25.32. If λ and μ are dominant weights, and α is a simple root with $\lambda(H_\alpha)$ and $\mu(H_\alpha)$ not zero, show that $\lambda + \mu - \alpha$ is a dominant weight and $\Gamma_\lambda \otimes \Gamma_\mu$ contains the irreducible representation $\Gamma_{\lambda+\mu-\alpha}$ with multiplicity one. So

$$\Gamma_\lambda \otimes \Gamma_\mu = \Gamma_{\lambda+\mu} \oplus \Gamma_{\lambda+\mu-\alpha} \oplus \text{others}.$$

In case $\mu = \lambda$, with $\lambda(H_\alpha) \neq 0$, $\mathrm{Sym}^2(\Gamma_\lambda)$ contains $\Gamma_{\lambda+\mu}$, while $\wedge^2(\Gamma_\lambda)$ contains $\Gamma_{\lambda+\mu-\alpha}$.

Exercise 25.33. If $\lambda + \zeta$ is a dominant weight for each weight ζ of Γ_μ, show that the irreducible representations appearing in $\Gamma_\lambda \otimes \Gamma_\mu$ are exactly the $\Gamma_{\lambda+\zeta}$. In fact, with no assumptions, every component of $\Gamma_\lambda \otimes \Gamma_\mu$ always has this form. One can show that $N_{\lambda\mu\nu}$ is the dimension of

$$\{v \in (\Gamma_\lambda)_{\nu-\mu} \colon H_i^{l_i+1}(v) = 0, 1 \le i \le n, l_i = \mu(H_i)\}.$$

For this, see [Žel, §131].

For other general formulas for the multiplicities $N_{\lambda\mu\nu}$ see [Kem], [K-N], [Li], and [Kum1], [Kum2].

We have seen in Exercise 6.12 a formula for decomposing the representation Γ_λ of $GL_m\mathbb{C}$ when restricted to the subgroup $GL_{m-1}\mathbb{C}$. In this case the multiplicities of the irreducible components again have a simple combinatorial description. There are similar formulas for other classical groups. In the literature, such formulas are often called "branching formulas," or "modification rules." We will just state the analogues of this formula for the symplectic and orthogonal cases:

For $\mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{so}_{2n+1}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, the restriction is

$$\operatorname{Res}^{\mathfrak{so}_{2n+1}\mathbb{C}}_{\mathfrak{so}_{2n}\mathbb{C}}(\Gamma_\lambda) = \bigoplus \Gamma_{\bar{\lambda}}, \tag{25.34}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$ with

$$\lambda_1 \geq \bar{\lambda}_1 \geq \lambda_2 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{n-1} \geq \lambda_n \geq |\bar{\lambda}_n|,$$

with the $\bar{\lambda}_i$ and λ_i simultaneously all integers or all half integers.

For $\mathfrak{so}_{2n-1}\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{so}_{2n}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq |\lambda_n|)$,

$$\operatorname{Res}^{\mathfrak{so}_{2n}\mathbb{C}}_{\mathfrak{so}_{2n-1}\mathbb{C}}(\Gamma_\lambda) = \bigoplus \Gamma_{\bar{\lambda}}, \tag{25.35}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1})$ with

$$\lambda_1 \geq \bar{\lambda}_1 \geq \lambda_2 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{n-1} \geq |\lambda_n|,$$

with the $\bar{\lambda}_i$ and λ_i simultaneously all integers or all half integers.

For $\mathfrak{sp}_{2n-2}\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, the restriction is

$$\operatorname{Res}^{\mathfrak{sp}_{2n}\mathbb{C}}_{\mathfrak{sp}_{2n-2}\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.36}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1})$ with $\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_{n-1} \geq 0$, and the multiplicity $N_{\lambda\bar{\lambda}}$ is the number of sequences $p_1, \ldots, p_n$ of integers satisfying

$$\lambda_1 \geq p_1 \geq \lambda_2 \geq p_2 \geq \cdots \geq \lambda_n \geq p_n \geq 0$$

and

$$p_1 \geq \bar{\lambda}_1 \geq p_2 \cdots \geq p_{n-1} \geq \bar{\lambda}_{n-1} \geq p_n.$$

As in the case of $GL_n\mathbb{C}$, these formulas are equivalent to identities among symmetric polynomials. The reader may enjoy trying to work them out from this point of view, cf. Exercise 23.43 and [Boe]. A less computational approach is given in [Žel].

As we saw in the case of the general linear group, these branching rules can be used inductively to compute the dimensions of the weight spaces. For example, for $\mathfrak{so}_m\mathbb{C}$ consider the chain

$$\mathfrak{so}_m\mathbb{C} \supset \mathfrak{so}_{m-1}\mathbb{C} \supset \mathfrak{so}_{m-2}\mathbb{C} \supset \cdots \supset \mathfrak{so}_3\mathbb{C}.$$

Decomposing a representation successively from one layer to the next will finally write it as a sum of one-dimensional weight spaces, and the dimension can be read off from the number of "partitions" in chains that start with the given λ. The representations can be constructed from these chains, as described by Gelfand and Zetlin, cf. [Žel, §10].

Similarly, one can ask for formulas for decomposing restrictions for other inclusions, such as the natural embeddings: $Sp_{2n}\mathbb{C} \subset SL_{2n}\mathbb{C}$, $SO_m\mathbb{C} \subset SL_m\mathbb{C}$, $GL_m\mathbb{C} \times GL_n\mathbb{C} \subset GL_{m+n}\mathbb{C}$, $GL_m\mathbb{C} \times GL_n\mathbb{C} \subset GL_{mn}\mathbb{C}$, $SL_n\mathbb{C} \subset Sp_{2n}\mathbb{C}$, $SL_n\mathbb{C} \subset SO_{2n+1}\mathbb{C}$, $SL_n\mathbb{C} \subset SO_{2n}\mathbb{C}$, to mention just a few. Such formulas are determined in principle by computing what happens to generators of the representation rings, which is not hard: one need only decompose exterior or symmetric products of standard representations, cf. Exercise 23.31. A few closed formulas for decomposing more general representations can also be found in the literature. We state what happens when the irreducible representations of $GL_m\mathbb{C}$ are restricted to the orthogonal or symplectic subgroups, referring to [Lit3] for the proofs:

For $O_m\mathbb{C} \subset GL_m\mathbb{C}$, with $m = 2n$ or $2n+1$, given $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$,

$$\mathrm{Res}_{O_m\mathbb{C}}^{GL_m\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.37}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_n \geq 0)$, where

$$N_{\lambda\bar{\lambda}} = \sum_\delta N_{\delta\bar{\lambda}\lambda},$$

with $N_{\delta\bar{\lambda}\lambda}$ the Littlewood–Richardson coefficient, and the sum over all $\delta = (\delta_1 \geq \delta_2 \geq \cdots)$ with all δ_i even.

Exercise 23.38. Show that the representation $\Gamma_{(2,2)}$ of $GL_m\mathbb{C}$ restricts to the direct sum

$$\Gamma_{(2,2)} \oplus \Gamma_{(2)} \oplus \Gamma_{(0)}$$

over $O_m\mathbb{C}$. (This decomposition is important in differential geometry: the *Riemann–Christoffel* tensor has type (2, 2), and the above three components of its decomposition are the *conformal curvature* tensor, the *Ricci* tensor, and the *scalar* curvature, respectively.)

Similarly for $Sp_{2n}\mathbb{C} \subset GL_{2n}\mathbb{C}$,

$$\mathrm{Res}_{Sp_{2n}\mathbb{C}}^{GL_{2n}\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.39}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_n \geq 0)$, where

$$N_{\lambda\bar{\lambda}} = \sum_\eta N_{\eta\bar{\lambda}\lambda},$$

$N_{\eta\bar{\lambda}\lambda}$ is the Littlewood–Richardson coefficient, and the sum is over all $\eta = (\eta_1 = \eta_2 \geq \eta_3 = \eta_4 \geq \cdots)$ with each part occurring an even number of times.

It is perhaps worth pointing out the the decomposition of tensor products is a special case of the decomposition of restrictions: the exterior tensor product $\Gamma_\lambda \boxtimes \Gamma_\mu$ of two irreducible representations of G is an irreducible representation of $G \times G$, and the restriction of this to the diagonal embedding of G in $G \times G$ is the usual tensor product $\Gamma_\lambda \otimes \Gamma_\mu$.

There are also some general formulas, valid whenever $\mathfrak{g}$ is a semisimple Lie subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Assume that the Cartan subalgebra $\bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{h}$, so we have a restriction from $\mathfrak{h}^*$ to $\bar{\mathfrak{h}}^*$, and we assume the half-spaces determining positive roots are compatible. We write $\bar{\mu}$ for weights of $\bar{\mathfrak{g}}$, and we write $\mu \downarrow \bar{\mu}$ to mean that a weight μ of $\mathfrak{g}$ restricts to $\bar{\mu}$. Similarly write $\overline{W}$ for a typical element of the Weyl group of $\bar{\mathfrak{g}}$, and $\bar{\rho}$ for half the sum of its positive weights. If λ (resp. $\bar{\lambda}$) is a dominant weight for $\mathfrak{g}$ (resp. $\bar{\mathfrak{g}}$), let $N_{\lambda\bar{\lambda}}$ denote the multiplicity with which $\Gamma_{\bar{\lambda}}$ appears in the restriction of Γ_λ to $\bar{\mathfrak{g}}$, i.e.,

$$\operatorname{Res}(\Gamma_\lambda) = \bigoplus_{\bar{\lambda}} N_{\lambda\bar{\lambda}} \Gamma_{\bar{\lambda}}.$$

Exercise 25.40*. Show that, for any dominant weight λ of $\mathfrak{g}$ and any weight $\bar{\mu}$ of $\bar{\mathfrak{g}}$,

$$\sum_{\mu \downarrow \bar{\mu}} n_\mu(\Gamma_\lambda) = \sum_{\bar{\lambda}} N_{\lambda\bar{\lambda}} n_{\bar{\mu}}(\Gamma_{\bar{\lambda}}).$$

Exercise 25.41* (Klimyk). Show that

$$N_{\lambda\bar{\lambda}} = \sum_{\overline{W}} (-1)^{\overline{W}} \sum_{\mu \downarrow \bar{\lambda} + \bar{\rho} - \overline{W}(\bar{\rho})} n_\mu(\Gamma_\lambda).$$

Exercise 25.42. Show that if the formula of the preceding exercise is applied to the diagonal embedding of $\mathfrak{g}$ in $\mathfrak{g} \times \mathfrak{g}$, then the Racah formula of Exercise 25.31 results.

For additional formulas of a similar vein, as well as discussions of how they can be implemented on a computer, there are several articles in *SIAM J. Appl. Math.* **25**, 1973.

Finally, we note that it is possible, for any semisimple Lie algebra $\mathfrak{g}$, to make the direct sum of all its irreducible representations into a commutative algebra, generalizing constructions we saw in Lectures 15, §17, and §19. Let $\Gamma_{\omega_1}, \ldots, \Gamma_{\omega_n}$ be the irreducible representations corresponding to the fundamental weights $\omega_1, \ldots, \omega_n$. Let

$$A^\bullet = \operatorname{Sym}^\bullet(\Gamma_{\omega_1} \oplus \cdots \oplus \Gamma_{\omega_n}).$$

This is a commutative graded algebra, the direct sum of pieces

$$A^{\mathbf{a}} = \bigoplus_{a_1, \ldots, a_n} \operatorname{Sym}^{a_1}(\Gamma_{\omega_1}) \otimes \cdots \otimes \operatorname{Sym}^{a_n}(\Gamma_{\omega_n}),$$

where $\mathbf{a} = (a_1, \ldots, a_n)$ is an n-tuple of non-negative integers. Then $A^{\mathbf{a}}$ is the direct sum of the irreducible representation Γ_λ whose highest weight is $\lambda = \sum a_i \omega_i$, and a sum $J^{\mathbf{a}}$ of representations whose highest weight is strictly smaller. As before, weight considerations show that $J^\bullet = \bigoplus_{\mathbf{a}} J^{\mathbf{a}}$ is an ideal in $A^\bullet$, so the quotient

$$A^\bullet/J^\bullet = \bigoplus_\lambda \Gamma_\lambda$$

is the direct sum of all the irreducible representations. The product

$$\Gamma_\lambda \otimes \Gamma_\mu \to \Gamma_{\lambda+\mu}$$

in this ring is often called *Cartan multiplication*; note that the fact that $\Gamma_{\lambda+\mu}$ occurs once in the tensor product determines such a projection, but only up to multiplication by a scalar.

Using ideas of §25.1, it is possible to give generators for the ideal $J^\bullet$. If C is the Casimir operator, we know that C acts on all representations and is multiplication by the constant $c_\lambda = (\lambda, \lambda) + (2\lambda, \rho)$ on the irreducible representation with highest weight λ. Therefore, if $\lambda = \sum a_i\omega_i$, the endomorphism $C - c_\lambda I$ of $A^\bullet$ vanishes on the factor Γ_λ, and on each of the representations Γ_μ of lower weight μ it is multiplication by $c_\mu - c_\lambda \neq 0$ [cf. (25.2)]. It follows that

$$J^\bullet = \operatorname{Image}(C - c_\lambda I\colon A^\bullet \to A^\bullet).$$

Exercise 25.43*. Write $C = \sum U_i U_i'$ as in §25.1. Show that for $v_1, \ldots, v_m$ vectors in the fundamental weight spaces, with $v_j \in \Gamma_{\alpha_j}$ and $\sum \alpha_j = \sum a_i\omega_i$, the element $(C - c_\lambda I)(v_1 \cdot v_2 \cdot \ldots \cdot v_m)$ is the sum over all pairs j, k, with $1 \leq j < k \leq m$, of the terms

$$\left(\sum_i (U_i(v_j)\cdot U_i'(v_k) + U_i'(v_j)\cdot U_i(v_k)) - 2(\alpha_j, \alpha_k)v_j\cdot v_k\right)\cdot \prod_{l\neq j,k} v_l.$$

From this exercise follows a theorem of Kostant: $J^\bullet$ is generated by the elements

$$\sum_i (U_i(v)\cdot U_i'(w) + U_i'(v)\cdot U_i(w)) - 2(\alpha, \beta)v\cdot w$$

for $v \in \Gamma_\alpha$, $w \in \Gamma_\beta$, with α and β fundamental roots. For the classical Lie algebras, this formula can be used to find concrete realizations of the ring. If one wants a similar ring for a semisimple Lie group, one has the same ring, of course, when the group is simply connected; this leads to the ring described in Lectures 15 and 17 for $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$. For $\mathrm{SO}_m\mathbb{C}$, little change is needed when m is odd, but there is more work for m even. Details can be found in [L-T].

LECTURE 26

Real Lie Algebras and Lie Groups

In this lecture we indicate how to complete the last step in the process outlined at the beginning of Part II: to take our knowledge of the classification and representation theory of complex algebras and groups and deduce the corresponding statements in the real case. We do this in the first section, giving a list of the simple classical real Lie algebras and saying a few words about the corresponding groups and their (complex) representations. The existence of a compact group whose Lie algebra has as complexification a given semisimple complex Lie algebra makes it possible to give another (indeed, the original) way to prove the Weyl character formula; we sketch this in §26.2. Finally, we can ask in regard to real Lie groups G a question analogous to one asked for the representations of finite groups in §3.5: which of the complex representations V of G actually come from real ones. We answer this in the most commonly encountered cases in §26.3. In this final lecture, proofs, when we attempt them, are generally only sketched and may require more than the usual fortitude from the reader.

§26.1. Classification of Real Simple Lie Algebras and Groups

Having described the semisimple complex Lie algebras, we now address the analogous problem for real Lie algebras. Since the complexification $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ of a semisimple real Lie algebra $\mathfrak{g}_0$ is a semisimple complex Lie algebra and we have classified those, we are reduced to the problem of describing the *real forms* of the complex semisimple Lie algebras: that is, for a given complex Lie algebra $\mathfrak{g}$, finding all real Lie algebras $\mathfrak{g}_0$ with

$$\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \underset{\iota}{\cong} \mathfrak{g}.$$

We saw many of the real forms of the classical complex Lie groups and algebras back in Lectures 7 and 8. In this section we will indicate one way to approach the question systematically, but we will only include sketches of proofs.

To get the idea of what to expect, let us work out real forms of $\mathfrak{sl}_2\mathbb{C}$ in detail. To do this, suppose $\mathfrak{g}_0$ is any real Lie subalgebra of $\mathfrak{sl}_2\mathbb{C}$, with $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2\mathbb{C}$. The natural thing to do is to try to carry out our analysis of semisimple Lie algebras for the real Lie algebra $\mathfrak{g}_0$: that is, find an element $H \in \mathfrak{g}_0$ such that $\operatorname{ad}(H)$ acts semisimply on $\mathfrak{g}_0$, decompose $\mathfrak{g}_0$ into eigenspaces, and so on. The first part of this presents no problem: since the subset of $\mathfrak{sl}_2\mathbb{C}$ of non-semisimple matrices is a proper algebraic subvariety, it cannot contain the real subspace $\mathfrak{g}_0 \subset \mathfrak{sl}_2\mathbb{C}$, so that we can certainly find a semisimple $H \in \mathfrak{g}_0$.

The next thing is to consider the eigenspaces of $\operatorname{ad}(H)$ acting on $\mathfrak{g}$. Of course, $\operatorname{ad}(H)$ has one eigenvalue 0, corresponding to the eigenspace $\mathfrak{h}_0 = \mathbb{R}\cdot H$ spanned by H. The remaining two eigenvalues must then sum to zero, which leaves just two possibilities:

(i) $\operatorname{ad}(H)$ has eigenvalues λ and $-\lambda$, for λ a nonzero real number; multiplying H by a real scalar, we can take $\lambda = 2$. In this case we obtain a decomposition of the vector space $\mathfrak{g}_0$ into one-dimensional eigenspaces

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}.$$

We can then choose $X \in \mathfrak{g}_2$ and $Y \in \mathfrak{g}_{-2}$; the standard argument then shows that the bracket $[X, Y]$ is a nonzero multiple of H, which we may take to be 1 by rechoosing X and Y. We thus have the real form $\mathfrak{sl}_2\mathbb{R}$, with the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(ii) $\operatorname{ad}(H)$ has eigenvalues $i\lambda$ and $-i\lambda$ for λ some nonzero real number; again, adjusting H by a real scalar we may take $\lambda = 1$. In this case, of course, there are no real eigenvectors for the action of $\operatorname{ad}(H)$ on $\mathfrak{g}_0$; but we can decompose $\mathfrak{g}_0$ into the direct sum of $\mathfrak{h}_0$ and the two-dimensional subspace $\mathfrak{g}_{\{i,-i\}}$ corresponding to the pair of eigenvalues i and $-i$. We may then choose a basis B and C for $\mathfrak{g}_{\{i,-i\}}$ with

$$[H, B] = C \quad \text{and} \quad [H, C] = -B.$$

The commutator $[B, C]$ will then be a nonzero multiple of H, which we may take to be either H or $-H$ (we can multiply B and C simultaneously by a scalar μ, which multiplies the commutator $[B, C]$ by μ^2). In the latter case, we see that $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}_2\mathbb{R}$ again: these are the relations we get if we take as basis for $\mathfrak{sl}_2\mathbb{C}$ the three vectors

$$H = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, if the commutator $[B, C] = H$, we do get a new example: $\mathfrak{g}_0$ is in this case isomorphic to the algebra

$$\mathfrak{su}_2 = \{A : {}^t\bar{A} = -A \text{ and } \operatorname{trace}(A) = 0\} \subset \mathfrak{sl}_2\mathbb{C},$$

which has as basis

$$H = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}.$$

Exercise 26.1. Carry out this analysis for the real Lie algebras $\mathfrak{so}_3\mathbb{R}$ and $\mathfrak{so}_{2,1}\mathbb{R}$. In particular, give an isomorphism of each with either $\mathfrak{sl}_2\mathbb{R}$ or $\mathfrak{su}_2$.

This completes our analysis of the real forms of $\mathfrak{sl}_2\mathbb{C}$. In the general case, we can try to apply a similar analysis, and indeed at least one aspect generalizes: given a real form $\mathfrak{g}_0 \subset \mathfrak{g}$ of the complex semisimple Lie algebra $\mathfrak{g}$, we can find a real subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ such that $\mathfrak{h}_0 \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$; this is called a *Cartan subalgebra* of $\mathfrak{g}_0$. There is a further complication in the case of Lie algebras of rank 2 or more: the values on $\mathfrak{h}_0$ of a root $\alpha \in R$ of $\mathfrak{g}$ need not be either all real or all purely imaginary. We, thus, need to consider the root spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{\bar{\alpha}}$, $\mathfrak{g}_{-\alpha}$, and $\mathfrak{g}_{-\bar{\alpha}}$, and the subalgebra they generate, at the same time. Moreover, as we saw in the above example, whether the values of the roots $\alpha \in R$ of $\mathfrak{g}$ on the real subspace $\mathfrak{h}_0$ are real, purely imaginary, or neither will in general depend on the choice of $\mathfrak{h}_0$.

Exercise 26.2*. In the case of $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{R} \subset \mathfrak{g} = \mathfrak{sl}_3\mathbb{C}$, suppose we choose as Cartan subalgebra $\mathfrak{h}_0$ the space spanned over $\mathbb{R}$ by the elements

$$H_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Show that this is indeed a Cartan subalgebra, and find the decomposition of $\mathfrak{g}$ into eigenspaces for the action of $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$. In particular, find the roots of $\mathfrak{g}$ as linear functions on $\mathfrak{h}$, and describe the corresponding decomposition of $\mathfrak{g}_0$.

Judging from these examples, it is probably prudent to resist the temptation to try to carry out an analysis of real semisimple Lie algebras via an analogue of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$ in this case. Rather, in the present book, we will do two things. First, we will give the statement of the classification theorem for the real forms of the classical algebras—that is, we will list all the simple real Lie algebras whose complexifications are classical algebras. Second, we will focus on two distinguished real forms possessed by any real semisimple Lie algebra, the *split form* and the *compact form*. These are the two forms that you see most often; and the existence of the latter in particular will be essential in the following section.

For the first, it turns out to be enough to work out the complexifications $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_0 \oplus i \cdot \mathfrak{g}_0$ of the real Lie algebras $\mathfrak{g}_0$ we know. The list is:

Real Lie algebra	Complexification
$\mathfrak{sl}_n\mathbb{R}$	$\mathfrak{sl}_n\mathbb{C}$
$\mathfrak{sl}_n\mathbb{C}$	$\mathfrak{sl}_n\mathbb{C} \times \mathfrak{sl}_n\mathbb{C}$
$\mathfrak{sl}_n\mathbb{H} = \mathfrak{gl}_n\mathbb{H}/\mathbb{R}$	$\mathfrak{sl}_{2n}\mathbb{C}$
$\mathfrak{so}_{p,q}\mathbb{R}$	$\mathfrak{so}_{p+q}\mathbb{C}$
$\mathfrak{so}_n\mathbb{C}$	$\mathfrak{so}_n\mathbb{C} \times \mathfrak{so}_n\mathbb{C}$
$\mathfrak{sp}_{2n}\mathbb{R}$	$\mathfrak{sp}_{2n}\mathbb{C}$
$\mathfrak{sp}_{2n}\mathbb{C}$	$\mathfrak{sp}_{2n}\mathbb{C} \times \mathfrak{sp}_{2n}\mathbb{C}$
$\mathfrak{su}_{p,q}$	$\mathfrak{sl}_{p+q}\mathbb{C}$
$\mathfrak{u}_{p,q}\mathbb{H}$	$\mathfrak{sp}_{2(p+q)}\mathbb{C}$
$\mathfrak{u}_n^*\mathbb{H}$	$\mathfrak{so}_{2n}\mathbb{C}$

The last two in the left-hand column are the Lie algebras of the groups $U_{p,q}\mathbb{H}$ and $U_n^*\mathbb{H}$ of automorphisms of a quaternionic vector space preserving a Hermitian form with signature (p, q), and a skew-symmetric Hermitian form, respectively.

We should first verify that the algebras on the right are indeed the complexifications of those on the left. Some are obvious, such as the complexification

$$(\mathfrak{sl}_n\mathbb{R})_{\mathbb{C}} = \mathfrak{sl}_n\mathbb{R} \oplus i \cdot \mathfrak{sl}_n\mathbb{R} = \mathfrak{sl}_n\mathbb{C}.$$

The same goes for $\mathfrak{so}_{p,q}\mathbb{R}$ and $\mathfrak{sp}_{2n}\mathbb{R}$.

Next, consider the complexification of

$$\mathfrak{su}_n = \{A \in \mathfrak{sl}_n\mathbb{C}: {}^t\bar{A} = -A\}.$$

To see that $\mathfrak{sl}_n\mathbb{C} = \mathfrak{su}_n \oplus i \cdot \mathfrak{su}_n$, let $M \in \mathfrak{sl}_n\mathbb{C}$, and write

$$M = \tfrac{1}{2}(M - {}^t\overline{M}) + \tfrac{1}{2}(M + {}^t\overline{M}) = \tfrac{1}{2}A + \tfrac{1}{2}B;$$

then $A \in \mathfrak{su}_n$, $iB \in \mathfrak{su}_n$, and $M = \frac{1}{2}A - i(i/2)B$.

The general case of $\mathfrak{su}_{p,q} \subset \mathfrak{sl}_{p+q}\mathbb{C}$ is similar: if the form is given by $(x, y) = {}^t\bar{x}Qy$, then $\mathfrak{su}_{p,q} = \{A: {}^t\bar{A}Q = -QA\}$. Writing $M \in \mathfrak{sl}_{p+q}\mathbb{C}$ in the form

$$M = \tfrac{1}{2}(M - Q \cdot {}^t\overline{M} \cdot Q) - i \cdot (\tfrac{1}{2}(iM + iQ \cdot {}^t\overline{M} \cdot Q))$$

and using $\bar{Q} = {}^tQ = Q^{-1} = Q$, one sees that $M \in \mathfrak{su}_{p,q} \oplus i \cdot \mathfrak{su}_{p,q}$.

For the complexification of $\mathfrak{sl}_m\mathbb{C}$, embed $\mathfrak{sl}_m\mathbb{C}$ in $\mathfrak{sl}_m\mathbb{C} \times \mathfrak{sl}_m\mathbb{C}$ by $A \mapsto (A, \bar{A})$. Given any pair (B, C), write

$$\begin{aligned}(B, C) &= \tfrac{1}{2}(B + \bar{C}, \bar{B} + C) + \tfrac{1}{2}(B - \bar{C}, -\bar{B} + C) \\ &= \tfrac{1}{2}(B + \bar{C}, \bar{B} + C) - i \cdot (\tfrac{1}{2}(iB + \overline{iC}, \overline{iB} + iC).\end{aligned}$$

For the quaternionic Lie algebra, from the description of $\mathrm{GL}_n\mathbb{H}$ we saw in Lecture 7, we have

$$\mathfrak{gl}_n\mathbb{H} = \{A \in \mathfrak{gl}_{2n}\mathbb{C}: AJ = J\bar{A}\},$$

with $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. As before, for $M \in \mathfrak{gl}_{2n}\mathbb{C}$, we can write

$$M = \tfrac{1}{2}(M - J \cdot \overline{M} \cdot J) - i \cdot (\tfrac{1}{2}(iM + iJ \cdot \overline{iM} \cdot J))$$

to see that $\mathfrak{gl}_n\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{gl}_{2n}\mathbb{C}$.

Exercise 26.3. Verify the rest of the list.

The theorem, which also goes back to Cartan, is that *this includes the complete list of simple real Lie algebras associated to the classical complex types* (A_n)–(D_n). In fact, there are an additional 17 simple real Lie algebras associated with the five exceptional Lie algebras. The proof of this theorem is rather long, and we refer to the literature (cf. [H-S], [Hel], [Ar]) for it.

Split Forms and Compact Forms

Rather than try to classify in general the real forms $\mathfrak{g}_0$ of a semisimple Lie algebra $\mathfrak{g}$, we would like to focus here on two particular forms that are possessed by every semisimple Lie algebra and that are by far the most commonly dealt with in practice: the *split form* and the *compact form.*

These represent the two extremes of behavior of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$ with respect to the real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$. To begin with, the *split form* of $\mathfrak{g}$ is a form $\mathfrak{g}_0$ such that there exists a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ (that is, a subalgebra whose complexification $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$) whose action on $\mathfrak{g}_0$ has all real eigenvalues—i.e., such that all the roots $\alpha \in R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}$) assume all real values on the subspace $\mathfrak{h}_0$. In this case we have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus (\bigoplus \mathfrak{i}_\alpha)$$

of $\mathfrak{g}_0$ into $\mathfrak{h}_0$ and one-dimensional eigenspaces $\mathfrak{i}_\alpha$ for the action of $\mathfrak{h}_0$ (each $\mathfrak{i}_\alpha$ will just be the intersection of the root space $\mathfrak{g}_\alpha \subset \mathfrak{g}$ with $\mathfrak{g}_0$); each pair $\mathfrak{i}_\alpha$ and $\mathfrak{i}_{-\alpha}$ will generate a subalgebra isomorphic to $\mathfrak{sl}_2\mathbb{R}$. As we will see momentarily, this uniquely characterizes the real form $\mathfrak{g}_0$ of $\mathfrak{g}$.

By contrast, in the *compact form* all the roots $\alpha \in R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}$) assume all purely imaginary values on the subspace $\mathfrak{h}_0$. We accordingly have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus (\bigoplus \mathfrak{l}_\alpha)$$

of $\mathfrak{g}_0$ into $\mathfrak{h}_0$ and two-dimensional spaces on which $\mathfrak{h}_0$ acts by rotation (each $\mathfrak{l}_\alpha$ will just be the intersection of the root space $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with $\mathfrak{g}_0$); each $\mathfrak{l}_\alpha$ will generate a subalgebra isomorphic to $\mathfrak{su}_2$.

The existence of the split form of a semisimple complex Lie algebra was already established in Lecture 21: one way to construct a real—even rational

—form $\mathfrak{g}_0$ of a semisimple Lie algebra $\mathfrak{g}$ is by starting with any generator X_{α_i} for the root space for each positive simple root α_i, completing it to standard basis X_{α_i}, Y_{α_i}, and $H_i = [X_{\alpha_i}, Y_{\alpha_i}]$ for the corresponding $\mathfrak{s}_{\alpha_i} = \mathfrak{sl}_2\mathbb{C}$, and taking $\mathfrak{g}_0$ to be the real subalgebra generated by these elements. Choosing a way to write each positive root as a sum of simple roots even determined a basis $\{H_i \in \mathfrak{h}, X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}\}$ for $\mathfrak{g}_0$, as in (21.20). The Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is the real span of these H_i. Note that once $\mathfrak{h}$ is fixed for $\mathfrak{g}$, the real subalgebra $\mathfrak{h}_0$ is uniquely determined as the span of the H_α for all roots α. The algebra $\mathfrak{g}_0$ is determined up to isomorphism; it is sometimes called the *natural* real form of $\mathfrak{g}$. Note that this also demonstrates the uniqueness of the split form: it is the only real form $\mathfrak{g}_0$ of $\mathfrak{g}$ that has a Cartan subalgebra $\mathfrak{h}_0$ acting on $\mathfrak{g}_0$ with all real eigenvalues.

As for the compact form of a semisimple Lie algebra, it owes much of its significance (as well as its name) to the last condition in

Proposition 26.4. *Suppose $\mathfrak{g}$ is any complex semisimple Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{g}$ a real form of $\mathfrak{g}$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$, $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$ the corresponding Cartan subalgebra of $\mathfrak{g}$. The following are equivalent:*

(i) *Each root $\alpha \in R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ assumes purely imaginary values on $\mathfrak{h}_0$, and for each root α the subalgebra of $\mathfrak{g}_0$ generated by the intersection $\mathfrak{l}_\alpha$ of $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ with $\mathfrak{g}_0$ is isomorphic to $\mathfrak{su}_2$;*
(ii) *The restriction to $\mathfrak{g}_0$ of the Killing form of $\mathfrak{g}$ is negative definite;*
(iii) *The real Lie group G_0 with Lie algebra $\mathfrak{g}_0$ is compact.*

In (iii), G_0 can be taken to be the adjoint form of $\mathfrak{g}_0$. However, a theorem of Weyl ensures that the fundamental group of any such G_0 is finite, so the condition is independent of the choice of G_0. Note also that, by the equivalence with (ii) and (iii), the condition (i) must be independent of the choice of Cartan subalgebra $\mathfrak{h}_0$. This is in contrast with the split case, where we require only that there exist a Cartan subalgebra whose action on $\mathfrak{g}$ has all real eigenvalues; as we saw in the case of $\mathfrak{sl}_2\mathbb{R}$, in the split case a different $\mathfrak{h}_0$ may have imaginary eigenvalues.

Proof. We start by showing that the first condition implies the second; this will follow from direct observation. To begin with, the value of the Killing form on $H \in \mathfrak{h}_0$ is visibly

$$B(H, H) = \sum (\alpha(H))^2 < 0.$$

Next, the subspaces $\mathfrak{l}_\alpha$ are orthogonal to one another with respect to B, so it remains only to verify $B(Z, Z) < 0$ for a general member $Z \in \mathfrak{l}_\alpha$. To do this, let X and Y be generators of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}$ respectively, chosen so as to form, together with their commutator $H = [X, Y]$ a standard basis for $\mathfrak{sl}_2\mathbb{C}$. By the analysis of real forms of $\mathfrak{sl}_2\mathbb{C}$ above, we may take as generators of the algebra generated by $\mathfrak{l}_\alpha$ the elements iH, $U = X - Y$ and $V = iX + iY$. If we set

$$Z = aU + bV = (a + ib)\cdot X + (-a + ib)\cdot Y,$$

then we have

$$\begin{aligned}\operatorname{ad}(Z) \circ \operatorname{ad}(Z) = (a + ib)^2 \operatorname{ad}(X) \circ \operatorname{ad}(X) \\ - (a^2 + b^2)(\operatorname{ad}(X) \circ \operatorname{ad}(Y) + \operatorname{ad}(Y) \circ \operatorname{ad}(X)) \\ + (a - ib)^2 \operatorname{ad}(Y) \circ \operatorname{ad}(Y).\end{aligned}$$

Now, $\operatorname{ad}(X) \circ \operatorname{ad}(X)$ and $\operatorname{ad}(Y) \circ \operatorname{ad}(Y)$ have no trace, so we can write

$$\operatorname{trace}(\operatorname{ad}(Z) \circ \operatorname{ad}(Z)) = -2 \cdot (a^2 + b^2) \cdot \operatorname{trace}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)). \tag{26.5}$$

By direct examination, in the representation $\operatorname{Sym}^n V$ of $\mathfrak{sl}_2\mathbb{C}$, $\operatorname{ad}(X) \circ \operatorname{ad}(Y)$ acts by multiplication by $(n - \lambda)(n + \lambda - 2)/4 \geq 0$ on the λ-eigenspace for H, from which we deduce that the right-hand side of (26.5) is negative.

Next, we show that the second condition implies the third. This is immediate: the adjoint form G_0 is the connected component of the identity of the group $\operatorname{Aut}(\mathfrak{g}_0)$. In particular, it is a closed subgroup of the adjoint group of $\mathfrak{g}$, and it acts faithfully on the real vector space $\mathfrak{g}_0$, preserving the bilinear form B. If B is negative definite it follows that G_0 is a closed subgroup of the orthogonal group $\mathrm{SO}_m\mathbb{R}$, which is compact.

Finally, if we know that G_0 is compact, by averaging we can construct a positive definite inner product on $\mathfrak{g}_0$ invariant under the action of G_0. For any X in $\mathfrak{g}_0$, $\operatorname{ad}(X)$ is represented by a skew-symmetric matrix $A = (a_{i,j})$ with respect to an orthonormal basis of $\mathfrak{g}_0$ (cf. (14.23)), so $B(X, X) = \operatorname{Tr}(A \circ A) = \sum_{i,j} a_{i,j}a_{j,i} = -\sum a_{i,j}^2 \leq 0$. In particular, the eigenvalues of $\operatorname{ad}(X)$ must be purely imaginary. Therefore $\alpha(\mathfrak{h}_0) \subset i\mathbb{R}$ and $\bar{\alpha} = -\alpha$ for any root α, from which (i) follows. □

We now claim that *every semisimple complex Lie algebra has a unique compact form.* To see this we need an algebraic notion which is, in fact, crucial to the classification theorem mentioned above: that of *conjugate linear involution.* If $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of a real Lie algebra $\mathfrak{g}_0$, there is a map $\sigma: \mathfrak{g} \to \mathfrak{g}$ which takes $x \otimes z$ to $x \otimes \bar{z}$ for $x \in \mathfrak{g}_0$ and $z \in \mathbb{C}$; it is conjugate linear, preserves Lie brackets, and σ^2 is the identity. The real algebra $\mathfrak{g}_0$ is the fixed subalgebra of σ, and conversely, given such a conjugate linear involution σ of a complex Lie algebra $\mathfrak{g}$, its fixed algebra $\mathfrak{g}^\sigma$ is a real form of $\mathfrak{g}$. To prove the claim, we start with the split, or natural form, as constructed in Lecture 21 and referred to above. With a basis for $\mathfrak{g}$ chosen as in this construction, it is not hard to show that there is a unique Lie algebra automorphism φ of $\mathfrak{g}$ that takes each element of $\mathfrak{h}$ to its negative and takes each X_α to Y_α (this follows from Claim 21.25). This automorphism φ is a complex linear involution which preserves the real subalgebra $\mathfrak{g}_0$. This automorphism commutes with the associated conjugate linear σ. The composite $\sigma\varphi = \varphi\sigma$ is a conjugate linear involution, from which it follows that its fixed part $\mathfrak{g}_c = \mathfrak{g}^{\sigma\varphi}$ is another real form of $\mathfrak{g}$. This has Cartan subalgebra $\mathfrak{h}_c = \mathfrak{h}^{\sigma\varphi} = i \cdot \mathfrak{h}_0$. We have seen that the restriction of the Killing form to $\mathfrak{h}_0$ is positive definite. It follows that its restriction to $\mathfrak{h}_c$ is negative definite, and hence that $\mathfrak{g}_c$ is a compact form of $\mathfrak{g}$. Finally, this construction of $\mathfrak{g}_c$ from $\mathfrak{g}_0$ is reversible, and from this one can deduce the uniqueness of the compact form.

We may see directly from this construction that

$$\mathfrak{g}_c = \mathfrak{h}_c \oplus \bigoplus_{\alpha \in R^+} \mathfrak{l}_\alpha,$$

where $\mathfrak{l}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})^{\sigma\varphi}$ is a real plane with $\mathfrak{l}_\alpha \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ and $[\mathfrak{h}_c, \mathfrak{l}_\alpha] \subset \mathfrak{l}_\alpha$.

Exercise 26.6. Verify that $\{A_j = i \cdot H_j : 1 \le j \le n\}$ is a basis for $\mathfrak{h}_c$, $\{B_\alpha = X_\alpha - Y_\alpha, C_\alpha = i \cdot (X_\alpha + Y_\alpha)\}$ is a basis for $\mathfrak{l}_\alpha$, and the action is given by

$$[A_j, B_\alpha] = p \cdot C_\alpha \quad \text{and} \quad [A_j, C_\alpha] = -p \cdot B_\alpha,$$

where p is the integer $\alpha(H_j)$. In particular, $\mathfrak{h}_c$ acts by rotations on the planes $\mathfrak{l}_\alpha$.

Our classical Lie algebras $\mathfrak{g}$ all came equipped with a natural real form $\mathfrak{g}_0$, and with a basis of the above type. These split forms are:

Complex simple Lie algebra	Split form
$\mathfrak{sl}_{n+1}\mathbb{C}$	$\mathfrak{sl}_{n+1}\mathbb{R}$
$\mathfrak{so}_{2n+1}\mathbb{C}$	$\mathfrak{so}_{n+1,n}$
$\mathfrak{sp}_{2n}\mathbb{C}$	$\mathfrak{sp}_{2n}\mathbb{R}$
$\mathfrak{so}_{2n}\mathbb{C}$	$\mathfrak{so}_{n,n}$

Exercise 26.7. For each of these split forms, find the corresponding compact form $\mathfrak{g}_c$.

Exercise 26.8. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra. Show that a subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is a Cartan subalgebra if and only if it is a maximal abelian subalgebra and the adjoint action on $\mathfrak{g}_0$ is semisimple.

Exercise 26.9*. Starting with a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ with associated conjugation σ, show that one can always find a compact form $\mathfrak{g}_c$ of $\mathfrak{g}$ such that $\sigma(\mathfrak{g}_c) = \mathfrak{g}_c$, and such that

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathfrak{g}_c$, and $\mathfrak{p} = \mathfrak{g}_0 \cap (i \cdot \mathfrak{g}_c)$. Such a decomposition is called a *Cartan decomposition* of $\mathfrak{g}_0$. It is unique up to inner automorphism.

Exercise 26.10*. For any real form $\mathfrak{g}_0$ of $\mathfrak{g}$, given by a conjugation σ, show that there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is preserved by σ, so $\mathfrak{g}_0 \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_0$.

Naturally, the various special isomorphisms between complex Lie algebras ($\mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C} \cong \mathfrak{sp}_2\mathbb{C}$, etc.) give rise to special isomorphisms among their real forms. For example, we have already seen that

$$\mathfrak{sl}_2\mathbb{R} \cong \mathfrak{su}_{1,1} \cong \mathfrak{so}_{2,1} \cong \mathfrak{sp}_2\mathbb{R},$$

while

$$\mathfrak{su}_2 \cong \mathfrak{so}_3\mathbb{R} \cong \mathfrak{sl}_1\mathbb{H} \cong \mathfrak{u}_1\mathbb{H}$$

(cf. Exercise 26.1). Similarly, each of the remaining three special isomorphisms of complex semisimple Lie algebras gives rise to isomorphisms between their real forms, as follows:

(i) $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$
 compact forms: $\mathfrak{so}_4\mathbb{R} \cong \mathfrak{su}_2 \times \mathfrak{su}_2$
 split forms: $\mathfrak{so}_{2,2} \cong \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R}$
 others: $\mathfrak{so}_{3,1} \cong \mathfrak{sl}_2\mathbb{C}$, $\mathfrak{u}_2^*\mathbb{H} \cong \mathfrak{su}_2 \times \mathfrak{sl}_2\mathbb{R}$.
(ii) $\mathfrak{sp}_4\mathbb{C} \cong \mathfrak{so}_5\mathbb{C}$
 compact forms: $\mathfrak{u}_2\mathbb{H} \cong \mathfrak{so}_5\mathbb{R}$
 split forms: $\mathfrak{sp}_4\mathbb{R} \cong \mathfrak{so}_{3,2}$
 other: $\mathfrak{u}_{1,1}\mathbb{H} \cong \mathfrak{so}_{4,1}$.
(iii) $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$
 compact forms: $\mathfrak{su}_4 \cong \mathfrak{so}_6\mathbb{R}$
 split forms: $\mathfrak{sl}_4\mathbb{R} \cong \mathfrak{so}_{3,3}$
 others: $\mathfrak{su}_{2,2} \cong \mathfrak{so}_{4,2}$; $\mathfrak{su}_{3,1} \cong \mathfrak{u}_3^*\mathbb{H}$; $\mathfrak{sl}_2\mathbb{H} \cong \mathfrak{so}_{5,1}$.

In addition, the extra automorphism of $\mathfrak{so}_8\mathbb{C}$ coming from triality gives rise to an isomorphism $\mathfrak{u}_4^*\mathbb{H} \cong \mathfrak{so}_{6,2}$.

Exercise 26.11. Verify some of the isomorphisms above. (Of course, in the case of compact and split forms, these are implied by the corresponding isomorphisms of complex Lie algebras, but it is worthwhile to see them directly in any case.)

Real Groups

We turn now to problem of describing the real Lie groups with these Lie algebras. Let G be the adjoint form of the semisimple complex Lie algebra $\mathfrak{g}$. If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, the associated conjugate linear involution σ of $\mathfrak{g}$ that fixes $\mathfrak{g}_0$ lifts to an involution $\tilde{\sigma}$ of G. (This follows from the functorial nature of the adjoint form, noting that G is regarded now as a real Lie group.) The fixed points $G^{\tilde{\sigma}}$ of this involution then form a closed subgroup of G; its connected component of the identity G_0 is a real Lie group whose Lie algebra is $\mathfrak{g}_0$. G is called the *complexification* of G_0.

We have seen in §23.1 that if $\Gamma = \Gamma_W$ is the lattice of those elements in $\mathfrak{h}$ on which all roots take integral values, then $2\pi i\Gamma$ is the kernel of the exponential mapping exp: $\mathfrak{h} \to G$ to the adjoint form. If $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, $T = \exp(\mathfrak{h}_0)$ will be compact precisely when the intersection of $\mathfrak{h}_0$ with the kernel $2\pi i\Gamma$ is a lattice of maximal rank. In this case, T will be a product of n copies of the circle S^1, $n = \dim(\mathfrak{h})$, and, since the Killing form on $\mathfrak{h}_0$ is negative definite, the corresponding real group G_0 will also be compact. Such a G_0 will be a maximal compact subgroup of G.

When $G_0 \subset G$ is a maximal compact subgroup, they have the same irreducible complex representations. Indeed, for any complex group G', each complex homomorphism from G to G' is the extension of a unique real homomorphism from G_0 to G'. This follows from the corresponding fact for Lie algebras and the fact that G_0 and G have the same fundamental group. This is another general fact, which implies the finiteness of the fundamental group of G_0; we omit the proof, noting only that it can be seen directly in the classical cases:

Exercise 26.12*. Prove that $\pi_1(G_0) \to \pi_1(G)$ is an isomorphism for each of the classical adjoint groups.

Exercise 26.13*. The special isomorphisms of real Lie algebras listed above give rise to special isomorphisms of real Lie groups. Can you find these?

It is another general fact that any compact (connected) Lie group is a quotient

$$(G_1 \times G_2 \times \cdots \times G_r \times T)/Z,$$

where the G_i are simple compact Lie groups, $T \cong (S^1)^k$ is a torus, and Z is a discrete subgroup of the center. In particular, its Lie algebra is the direct sum of a semisimple compact Lie algebra and an abelian Lie algebra. This provides another reason why the classification of irreducible representations in the real compact case and the semisimple complex case are essentially the same.

Representations of Real Lie Algebras

Finally, we should say a word here about the irreducible representations (always here in complex vector spaces!) of simple real Lie algebras. In some cases these are easily described in terms of the complex case: for example, the irreducible representations of $\mathfrak{su}_m$ or $\mathfrak{sl}_m\mathbb{R}$ are the same as those for $\mathfrak{sl}_m\mathbb{C}$, i.e., they are the restrictions of the irreducible representations $\Gamma_\lambda = \mathbb{S}_\lambda\mathbb{C}^m$ corresponding to partitions or Young diagrams λ. This is the situation in general whenever the complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ of the real Lie algebra $\mathfrak{g}_0$ is still simple: the representations of $\mathfrak{g}_0$ on complex vector spaces are exactly the representations of $\mathfrak{g}$. The situation is slightly different when we have a simple real Lie algebra whose complexification is not simple: for example, the irreducible representations of $\mathfrak{sl}_m\mathbb{C}$, regarded as a real Lie algebra, are of the form $\Gamma_\lambda \otimes \overline{\Gamma}_\mu$, where $\overline{\Gamma}_\mu$ is the conjugate representation of Γ_μ. The situation in general is expressed in the following

Exercise 26.14. Show that if $\mathfrak{g}_0$ is a simple real Lie algebra whose complexification $\mathfrak{g}$ is simple, its irreducible representations are the restrictions of (uniquely determined) irreducible representations of $\mathfrak{g}$. If $\mathfrak{g}_0$ is the underlying real algebra of a simple complex Lie algebra, show that the irreducible representations of $\mathfrak{g}_0$ are of the form $V \otimes \overline{W}$, where V and W are (uniquely determined) irreducible representations of the complex Lie algebra.